

HALL ALGEBRAS OF CYCLIC QUIVERS AND q -DEFORMED FOCK SPACES

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ABSTRACT. Based on the work of Ringel and Green, one can define the (Drinfeld) double Ringel–Hall algebra $\mathcal{D}(Q)$ of a quiver Q as well as its highest weight modules. The main purpose of the present paper is to show that the basic representation $L(\Lambda_0)$ of $\mathcal{D}(\Delta_n)$ of the cyclic quiver Δ_n provides a realization of the q -deformed Fock space \bigwedge^∞ defined by Hayashi. This is worked out by extending a construction of Varagnolo and Vasserot. By analysing the structure of nilpotent representations of Δ_n , we obtain a decomposition of the basic representation $L(\Lambda_0)$ which induces the Kashiwara–Miwa–Stern decomposition of \bigwedge^∞ and a construction of the canonical basis of \bigwedge^∞ defined by Leclerc and Thibon in terms of certain monomial basis elements in $\mathcal{D}(\Delta_n)$.

1. INTRODUCTION

In [39], Ringel introduced the Hall algebra $\mathcal{H}(\Delta_n)$ of the cyclic quiver Δ_n with n vertices and showed that its subalgebra generated by simple representations, called the composition algebra, is isomorphic to the positive part $\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n)$ of the quantized enveloping algebra $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$. Schiffmann [40] further showed that $\mathcal{H}(\Delta_n)$ is the tensor product of $\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n)$ with a central subalgebra which is the polynomial ring in infinitely many indeterminates. Following the approach in [44], the double Ringel–Hall algebra $\mathcal{D}(\Delta_n)$ was defined in [6]. Based on [12, 21] and an explicit description of central elements of $\mathcal{H}(\Delta_n)$ in [19], it was shown in [6, Th. 2.3.3] that $\mathcal{D}(\Delta_n)$ is isomorphic to the quantum affine algebra $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$ defined by Drinfeld’s new presentation [10].

The q -deformed Fock space representation \bigwedge^∞ of the quantized enveloping algebra $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ has been constructed by Hayashi [17], and its crystal basis was described by Misra and Miwa [36]. Further, by work of Kashiwara, Miwa, and Stern [27], the action of $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ on \bigwedge^∞ is centralized by a Heisenberg algebra which arises from affine Hecke algebras. This yields a bimodule isomorphism from \bigwedge^∞ to the tensor product of the basic representation of $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ and the Fock space representation of the Heisenberg algebra.

By defining a natural semilinear involution on \bigwedge^∞ , Leclerc and Thibon [29] obtained in an elementary way a canonical basis of \bigwedge^∞ . It was conjectured in [28, 29] that for $q = 1$, the coefficients of the transition matrix of the canonical basis on the natural basis of \bigwedge^∞ are equal to the decomposition numbers for Hecke algebras and quantum Schur algebras at roots of unity. These conjecture have been proved, respectively, by Ariki [1] and Varagnolo and Vasserot [45]. For the categorification of the Fock space, see, for example, [42, 18, 43].

In [45], Varagnolo and Vasserot extended the $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ -action on the Fock space \bigwedge^∞ to that of the extended Ringel–Hall algebra $\mathcal{D}(\Delta_n)^{\leq 0}$ of the cyclic quiver Δ_n . They also showed that the canonical basis of the Ringel–Hall algebra $\mathcal{H}(\Delta_n)$ in the sense of Lusztig induces a basis of \bigwedge^∞ which conjecturally coincides with the canonical basis constructed by Leclerc–Thibon [29]. This conjecture was proved by Schiffmann [40] by identifying the central subalgebra of $\mathcal{H}(\Delta_n)$ with the ring of symmetric functions.

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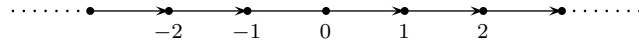
The main purpose of the present paper is to extend Varagnolo–Vasserot’s construction to obtain a $\mathcal{D}(\Delta_n)$ -module structure on the Fock space \bigwedge^∞ which is shown to be isomorphic to the basic representation $L(\Lambda_0)$ of $\mathcal{D}(\Delta_n)$. Moreover, the central elements in the positive and negative parts of $\mathcal{D}(\Delta_n)$ constructed by Hubery [19] give rise naturally to the operators introduced in [27] which generate the Heisenberg algebra. Furthermore, the structure of $\mathcal{D}(\Delta_n)$ yields a decomposition of $L(\Lambda_0)$ which induces the Kashiwara–Miwa–Stern decomposition of \bigwedge^∞ . This also provides a way to construct the canonical basis of \bigwedge^∞ in [29] in terms of certain monomial basis elements of $\mathcal{D}(\Delta_n)$.

The paper is organized as follows. In Section 2 we review the classification of (nilpotent) representations of both infinite linear quiver Δ_∞ and the cyclic quiver Δ_n with n vertices and discuss their generic extensions. Section 3 recalls the definition of Ringel–Hall algebras $\mathcal{H}(\Delta_\infty)$ and $\mathcal{H}(\Delta_n)$ of Δ_∞ and Δ_n as well as the maps from the homogeneous spaces of $\mathcal{H}(\Delta_n)$ to those of $\mathcal{H}(\Delta_\infty)$ introduced in [45]. The images of basis elements of $\mathcal{H}(\Delta_n)$ under these maps are described. In Section 4 we first follow the approach in [44] to present the construction of double Ringel–Hall algebras of both Δ_∞ and Δ_n and then study the irreducible highest weight $\mathcal{D}(\Delta_n)$ -modules based on the results in [23]. Section 5 recalls from [17, 36, 45] the Fock space representation \bigwedge^∞ over $U_v(\widehat{\mathfrak{sl}}_\infty)$ ($\cong \mathcal{D}(\Delta_\infty)$) as well as over $U_v^+(\widehat{\mathfrak{sl}}_n)$. In Section 6 we define the $\mathcal{D}(\Delta_n)$ -module structure on \bigwedge^∞ based on [27, 45]. It is shown in Section 7 that \bigwedge^∞ is isomorphic to the basic representation of $\mathcal{D}(\Delta_n)$. In the final section, we present a way to construct the canonical basis of \bigwedge^∞ and interpret the “ladder method” construction of certain basis elements in \bigwedge^∞ in terms of generic extensions of nilpotent representations of Δ_n .

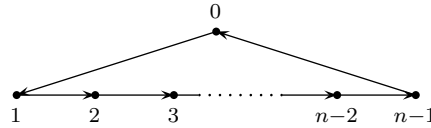
2. NILPOTENT REPRESENTATIONS AND GENERIC EXTENSIONS

In this section we consider nilpotent representations of both a cyclic quiver $\Delta = \Delta_n$ with n vertices ($n \geq 2$) and the infinite quiver $\Delta = \Delta_\infty$ of type A_∞^∞ and study their generic extensions. We show that the degeneration order of nilpotent representations of Δ_n induces the dominant order of partitions.

Let Δ_∞ denote the infinite quiver of type A_∞^∞



with vertex set $I = I_\infty = \mathbb{Z}$, and for $n \geq 2$, let Δ_n denote the cyclic quiver



with vertex set $I = I_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$. For each $i \in I_\infty = \mathbb{Z}$, let \bar{i} denote its residue class in $I_n = \mathbb{Z}/n\mathbb{Z}$. We also simply write $\bar{i} \pm 1$ to denote the residue class of $i \pm 1$ in $\mathbb{Z}/n\mathbb{Z}$.

Given a field k , we denote by $\text{Rep}^0 \Delta$ the category of finite dimensional nilpotent representations of Δ ($= \Delta_\infty$ or Δ_n) over k . (Note that each finite dimensional representation of Δ_∞ is automatically nilpotent.) Given a representation $V = (V_i, V_\rho) \in \text{Rep}^0 \Delta$, the vector $\mathbf{dim} V = (\dim_k V_i)_{i \in I}$ is called the *dimension vector* of V . The Grothendieck group of $\text{Rep}^0 \Delta$ is identified with the free abelian group $\mathbb{Z}I$ with basis I . Let $\{\varepsilon_i \mid i \in I\}$ denote the standard basis of $\mathbb{Z}I$. Thus, elements in $\mathbb{Z}I$ will be written as $\mathbf{d} = (d_i)_{i \in I}$ or $\mathbf{d} = \sum_{i \in I} d_i \varepsilon_i$. In case $I = \mathbb{Z}/n\mathbb{Z}$, we sometimes write \mathbb{Z}^n for $\mathbb{Z}I$.

The Euler form $\langle -, - \rangle : \mathbb{Z}I \times \mathbb{Z}I \rightarrow \mathbb{Z}$ is defined by

$$\langle \mathbf{dim} M, \mathbf{dim} N \rangle = \dim_k \text{Hom}_{k\Delta}(M, N) - \dim_k \text{Ext}_{k\Delta}^1(M, N).$$

Its symmetrization

$$(\mathbf{dim} M, \mathbf{dim} N) = \langle \mathbf{dim} M, \mathbf{dim} N \rangle + \langle \mathbf{dim} N, \mathbf{dim} M \rangle$$

is called the symmetric Euler form.

It is well known that the isoclasses of representations in $\text{Rep}^0 \Delta$ are parametrized by the set \mathfrak{M} consisting of all multisegments

$$\mathbf{m} = \sum_{i \in I, l \geq 1} m_{i,l} [i, l],$$

where all $m_{i,l} \in \mathbb{N}$, but finitely many, are zero. More precisely, the representation $M(\mathbf{m}) = M_k(\mathbf{m})$ associated with \mathbf{m} is defined by

$$M(\mathbf{m}) = \bigoplus_{i \in I, l \geq 1} m_{i,l} S_i[l],$$

where $S_i[l]$ denotes the representation of Δ with the simple top S_i and length l . For each $\mathbf{d} \in \mathbb{N}I$, put

$$\mathfrak{M}^{\mathbf{d}} = \{\mathbf{m} \in \mathfrak{M} \mid \mathbf{dim} M(\mathbf{m}) = \mathbf{d}\}.$$

Furthermore, we will write $\mathfrak{M} = \mathfrak{M}_\infty$ (resp., $\mathfrak{M} = \mathfrak{M}_n$) if $I = \mathbb{Z}$ (resp., $I = \mathbb{Z}/n\mathbb{Z}$).

It is also known that there exist Auslander–Reiten sequences in $\text{Rep}^0 \Delta$, that is, for each $M \in \text{Rep}^0 \Delta$, there is an Auslander–Reiten sequence

$$0 \longrightarrow \tau M \longrightarrow E \longrightarrow M \longrightarrow 0,$$

where τM denotes the Auslander–Reiten translation of M . It is clear that τ induces an isomorphism $\tau : \mathbb{Z}I \rightarrow \mathbb{Z}I$ such that $\tau(\mathbf{dim} M) = \mathbf{dim} \tau M$. In particular, $\tau(\varepsilon_i) = \varepsilon_{i+1}$, $\forall i \in I$. If $\Delta = \Delta_n$, then $\tau^{sn} = \text{id}$ for all $s \in \mathbb{Z}$. For $\mathbf{m} \in \mathfrak{M}$, let $\tau \mathbf{m}$ be defined by $M(\tau \mathbf{m}) \cong \tau M(\mathbf{m})$.

Given $\mathbf{d} \in \mathbb{N}I$, let $V = \bigoplus_{i \in I} V_i$ be an I -graded vector space with dimension vector \mathbf{d} . Consider

$$E_V = \{(x_i) \in \bigoplus_{i \in I} \text{Hom}_k(V_i, V_{i+1}) \mid x_{n-1} \cdots x_0 \text{ is nilpotent if } \Delta = \Delta_n\}.$$

Then each element $x \in E_V$ defines a representation (V, x) of dimension vector \mathbf{d} in $\text{Rep}^0 \Delta$. Moreover, the group

$$G_V = \prod_{i \in I} \text{GL}(V_i)$$

acts on E_V by conjugation, and there is a bijection between the G_V -orbits and the isoclasses of representations in $\text{Rep}^0 \Delta$ of dimension vector \mathbf{d} . For each $x \in E_V$, by \mathcal{O}_x we denote the G_V -orbit of x . In case k is algebraically closed, we have the equalities

$$(2.0.1) \quad \dim \mathcal{O}_x = \dim G_V - \dim \text{End}_{k\Delta}(V, x) = \sum_{i \in I} d_i^2 - \dim \text{End}_{k\Delta}(V, x).$$

By abuse of notation, for each $M \in \text{Rep}^0 \Delta$, we denote by \mathcal{O}_M the orbit of M .

Following [3, 37, 5], given two representations M, N in $\text{Rep}^0 \Delta$, there exists a unique (up to isomorphism) extension G of M by N such that $\dim \text{End}_{k\Delta}(G)$ is minimal. The extension G is called the *generic extension* of M by N , denoted by $M * N$. Moreover, generic extensions satisfy the associativity, i.e., for $L, M, N \in \text{Rep}^0 \Delta$,

$$L * (M * N) \cong (L * M) * N.$$

Let $\mathcal{M}(\Delta)$ denote the set of isoclasses of representations in $\text{Rep}^0 \Delta$. Define a multiplication on $\mathcal{M}(\Delta)$ by setting

$$[M] * [N] = [M * N].$$

Then $\mathcal{M}(\Delta)$ is a monoid with identity $[0]$, the isoclass of zero representation of Δ .

By [37, 5], the generic extension $M * N$ can be also characterized as the unique maximal element among all the extensions of M by N with respect to the degeneration order \leq_{deg} which is defined by setting $M \leq_{\text{deg}} N$ if $\mathbf{dim} M = \mathbf{dim} N$ and

$$(2.0.2) \quad \dim_k \text{Hom}_{k\Delta}(M, X) \geq \dim_k \text{Hom}_{k\Delta}(N, X), \quad \text{for all } X \in \text{Rep}^0 \Delta.$$

If k is algebraically closed, then $M \leq_{\text{deg}} N$ if and only if $\overline{\mathcal{O}}_M \subseteq \mathcal{O}_N$, where $\overline{\mathcal{O}}_M$ is the closure of \mathcal{O}_M . This defines a partial order relation on the set $\mathcal{M}(\Delta)$ of isoclasses of representations in $\text{Rep}^0 \Delta$; see [46, Th. 2] or [5, Lem. 3.2]. By [37, 2.4], for $M, N, M', N' \in \text{Rep}^0 \Delta$,

$$M' \leq_{\text{deg}} M, N' \leq_{\text{deg}} N \implies M' * N' \leq_{\text{deg}} M * N.$$

For $\mathbf{m}, \mathbf{m}' \in \mathfrak{M}_n$ (resp., \mathfrak{M}_∞), we write $\mathbf{m} \leq_{\text{deg}} \mathbf{m}'$ (resp., $\mathbf{m} \leq_{\text{deg}}^\infty \mathbf{m}'$) if $M(\mathbf{m}) \leq_{\text{deg}} M(\mathbf{m}')$ in $\text{Rep}^0 \Delta_n$ (resp., $\text{Rep} \Delta_\infty$).

By [4, 13], there is a covering functor

$$\mathcal{F} : \text{Rep} \Delta_\infty \longrightarrow \text{Rep}^0 \Delta_n$$

sending $S_i[l]$ to $S_i[l]$ for $i \in \mathbb{Z}$ and $l \geq 1$. Moreover, \mathcal{F} is dense and exact, and the Galois group of \mathcal{F} is the infinite cyclic group G generated by τ^n , i.e., $\tau^n(S_i[l] = S_{i+n}[l])$. For $\mathbf{m} \in \mathfrak{M}_\infty$, let $\mathcal{F}(\mathbf{m}) \in \mathfrak{M}_n$ be such that $M(\mathcal{F}(\mathbf{m})) \cong \mathcal{F}(M(\mathbf{m})) \in \text{Rep}^0 \Delta_n$. From (2.0.2) we easily deduce that for $M, N \in \text{Rep} \Delta_\infty$,

$$(2.0.3) \quad M \leq_{\text{deg}} N \implies \mathcal{F}(M) \leq_{\text{deg}} \mathcal{F}(N).$$

The following two classes of representations will play an important role later on. For each $\mathbf{d} = (d_i) \in \mathbb{N}I$, we set

$$S_{\mathbf{d}} = \bigoplus_{i \in I} d_i S_i[1] \in \text{Rep}^0 \Delta.$$

In other words, $S_{\mathbf{d}}$ is the unique semisimple representation of dimension vector \mathbf{d} .

Let Π be the set of all partitions $\lambda = (\lambda_1, \dots, \lambda_t)$ (i.e., $\lambda_1 \geq \dots \geq \lambda_t \geq 1$). For each $\lambda \in \Pi$, define

$$\mathbf{m}_\lambda = \sum_{s=1}^t [1 - s, \lambda_s] \in \mathfrak{M}.$$

Then

$$M(\mathbf{m}_\lambda) = S_0[\lambda_1] \oplus S_{-1}[\lambda_2] \oplus \dots \oplus S_{1-t}[\lambda_t] \in \text{Rep}^0 \Delta.$$

If $\Delta = \Delta_\infty$, then we sometimes write $\mathbf{m}_\lambda = \mathbf{m}_\lambda^\infty \in \mathfrak{M}_\infty$ to make a distinction. It follows from the definition that $\mathcal{F}(\mathbf{m}_\lambda^\infty) = \mathbf{m}_\lambda$ for all $\lambda \in \Pi$.

Proposition 2.1. *Let $\lambda, \mu \in \Pi$.*

(1) *If $\Delta = \Delta_\infty$, then*

$$\mathbf{dim} M(\mathbf{m}_\mu^\infty) = \mathbf{dim} M(\mathbf{m}_\lambda^\infty) \iff \mu = \lambda.$$

In particular, for each $\mathbf{m} \in \mathfrak{M}_\infty$, there exists at most one $\nu \in \Pi$ such that $\mathbf{m} = \mathbf{m}_\nu^\infty$.

(2) *If $\Delta = \Delta_n$, then*

$$M(\mathbf{m}_\mu) \leq_{\text{deg}} M(\mathbf{m}_\lambda) \implies \mu \leq \lambda,$$

where \leq is the dominance order on Π , i.e., $\mu \leq \lambda \iff \sum_{j=1}^i \mu_j \leq \sum_{j=1}^i \lambda_j, \forall i \geq 1$.

Proof. (1) By definition, both the socles of $M(\mathbf{m}_\lambda^\infty)$ and $M(\mathbf{m}_\mu^\infty)$ are multiplicity-free. Thus, comparing the socles of $S_0[\lambda_1]$ and $S_0[\mu_1]$ gives $\lambda_1 = \mu_1$. The lemma then follows from an inductive argument.

(2) Suppose $M(\mathbf{m}_\mu) \leq_{\deg} M(\mathbf{m}_\lambda)$. By viewing \mathbf{m}_λ and \mathbf{m}_μ as multipartitions in \mathfrak{M}_n , we obtain by [7, Prop. 2.7] that for each $l \geq 1$,

$$\sum_{s=1}^l \tilde{\mu}_s \geq \sum_{s=1}^l \tilde{\lambda}_s,$$

where $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots)$ and $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \dots)$ are the dual partition of λ and μ , respectively, that is, $\tilde{\mu} \supseteq \tilde{\lambda}$. By [35, 1.1], $\mu \leq \lambda$. \square

3. RINGEL–HALL ALGEBRA OF THE QUIVER Δ

In this section we introduce the Ringel–Hall algebra $\mathcal{H}(\Delta)$ of Δ ($= \Delta_n$ or Δ_∞) and the maps from homogeneous subspaces of $\mathcal{H}(\Delta_n)$ to those of $\mathcal{H}(\Delta_\infty)$ defined in [45, 6.1]. We also describe the images of basis elements of $\mathcal{H}(\Delta_n)$ under these maps.

The cyclic quiver Δ_n gives the $n \times n$ Cartan matrix $C_n = (a_{ij})_{i,j \in I}$ of type \hat{A}_{n-1} , while Δ_∞ defines the infinite Cartan matrix $C_\infty = (a_{ij})_{i,j \in \mathbb{Z}}$. Thus, we have the associated quantum enveloping algebras $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ and $\mathbf{U}_v(\widehat{\mathfrak{sl}}_\infty)$ which are $\mathbb{Q}(v)$ -algebras with generators $K_i^{\pm 1}, E_i, F_i, D$ ($i \in I = \mathbb{Z}/\mathbb{Z}_n$) and $K_i^{\pm 1}, E_i, F_i$ ($i \in \mathbb{Z}$), respectively, and the quantum Serre relations. In particular, the relations involving the generator D in $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ are

$$DD^{-1} = 1 = D^{-1}D, K_i D = D K_i, D E_i = v^{\delta_{0,i}} E_i D, D F_i = v^{-\delta_{0,i}} F_i D, \quad \forall i \in I;$$

see [2, Def. 3.16]. The subalgebra of $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ generated by $K_i^{\pm 1}, E_i, F_i$ ($i \in I = \mathbb{Z}/\mathbb{Z}_n$) is denoted by $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$.

By [38, 39, 16], for $\mathbf{p}, \mathbf{m}_1, \dots, \mathbf{m}_t \in \mathfrak{M}$, there is a polynomial $\varphi_{\mathbf{m}_1, \dots, \mathbf{m}_t}^{\mathbf{p}}(q) \in \mathbb{Z}[q]$ (called Hall polynomial) such that for each finite field k ,

$$\varphi_{\mathbf{m}_1, \dots, \mathbf{m}_t}^{\mathbf{p}}(|k|) = F_{M_k(\mathbf{m}_1), \dots, M_k(\mathbf{m}_t)}^{M_k(\mathbf{p})},$$

which is by definition the number of the filtrations

$$M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_{t-1} \supseteq M_t = 0$$

such that $M_{s-1}/M_s \cong M_k(\mathbf{m}_s)$ for all $1 \leq s \leq t$. It is also known that for each $\mathbf{m} \in \mathfrak{M}$, there is a polynomial $a_{\mathbf{m}}(q) \in \mathbb{Z}[q]$ such that for each finite field k ,

$$a_{\mathbf{m}}(|k|) = |\text{Aut}_{k\Delta}(M_k(\mathbf{m}))|.$$

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ be the Laurent polynomial ring over \mathbb{Z} in indeterminate v . By definition, the (twisted generic) *Ringel–Hall algebra* $\mathcal{H}(\Delta)$ of Δ is the free \mathcal{Z} -module with basis $\{u_{\mathbf{m}} | \mathbf{m} \in \mathfrak{M}\}$ and multiplication given by

$$(3.0.1) \quad u_{\mathbf{m}} u_{\mathbf{m}'} = v^{\langle \dim M(\mathbf{m}), \dim M(\mathbf{m}') \rangle} \sum_{\mathbf{p} \in \mathfrak{M}} \varphi_{\mathbf{m}, \mathbf{m}'}^{\mathbf{p}}(v^2) u_{\mathbf{p}}.$$

In practice, we also write $u_{\mathbf{m}} = u_{[M(\mathbf{m})]}$ in order to make certain calculations in terms of modules. Furthermore, for each $\mathbf{d} \in \mathbb{N}I$, we simply write $u_{\mathbf{d}} = u_{[S_{\mathbf{d}}]}$. Moreover, both $\mathcal{H}(\Delta)$ and $\mathcal{C}(\Delta)$ are $\mathbb{N}I$ -graded:

$$(3.0.2) \quad \mathcal{H}(\Delta) = \bigoplus_{\mathbf{d} \in \mathbb{N}I} \mathcal{H}(\Delta)_{\mathbf{d}} \quad \text{and} \quad \mathcal{C}(\Delta) = \bigoplus_{\mathbf{d} \in \mathbb{N}I} \mathcal{C}(\Delta)_{\mathbf{d}},$$

where $\mathcal{H}(\Delta)_{\mathbf{d}}$ is spanned by all $u_{\mathbf{m}}$ with $\mathbf{m} \in \mathfrak{M}^{\mathbf{d}}$ and $\mathcal{C}(\Delta)_{\mathbf{d}} = \mathcal{C}(\Delta) \cap \mathcal{H}(\Delta)_{\mathbf{d}}$. Since the Auslander–Reiten translate $\tau : \text{Rep}^0 \Delta \rightarrow \text{Rep}^0 \Delta$ is an auto-equivalence, it induces an automorphism $\tau : \mathcal{H}(\Delta) \rightarrow \mathcal{H}(\Delta)$, $u_{\mathbf{m}} \mapsto u_{\tau \mathbf{m}}$. We also consider the $\mathbb{Q}(v)$ -algebra

$$\mathcal{H}(\Delta) = \mathcal{H}(\Delta) \otimes_{\mathcal{Z}} \mathbb{Q}(v).$$

Remark 3.1. We remark that the Hall algebra of Δ defined in [45] is the opposite algebra of $\mathcal{H}(\Delta)$ given here with v being replaced by v^{-1} . Thus, v and v^{-1} should be swapped when comparing with the formulas in [45].

For each $i \in I$, set $u_i = u_{[S_i]}$. We then denote by $\mathcal{C}(\Delta)$ the subalgebra of $\mathcal{H}(\Delta)$ generated by the divided power $u_i^{(t)} = u_i^t/[t]!$, $i \in I$ and $t \geq 1$, called the *composition algebra* of Δ , where $[t]! = [t][t-1] \cdots [1]$ with $[m] = (v^m - v^{-m})/(v - v^{-1})$. It is known that $\mathcal{C}(\Delta_\infty) = \mathcal{H}(\Delta_\infty)$ and there is an isomorphism $\mathbf{U}_v^+(\mathfrak{sl}_\infty) \cong \mathcal{H}(\Delta_\infty)$ taking $E_i \mapsto u_i$, $\forall i \in I_\infty = \mathbb{Z}$. But, for $n \geq 2$, $\mathcal{C}(\Delta_n)$ is a proper subalgebra of $\mathcal{H}(\Delta_n)$. By [39],

$$\mathbf{U}_v^+(\widehat{\mathfrak{sl}}_n) \cong \mathcal{C}(\Delta_n) := \mathcal{C}(\Delta_n) \otimes_{\mathcal{Z}} \mathbb{Q}(v), \quad E_i \mapsto u_i, \quad \forall i \in I_n.$$

By [40, Th. 2.2], $\mathcal{H}(\Delta_n)$ is decomposed into the tensor product of $\mathcal{C}(\Delta_n)$ and a polynomial ring in infinitely many indeterminates which are central elements in $\mathcal{H}(\Delta_n)$. Such central elements have been explicitly constructed in [19]. More precisely, for each $t \geq 1$, let

$$(3.1.1) \quad c_t = (-1)^t v^{-2nt} \sum_{\mathbf{m}} (-1)^{\dim \text{End}(M(\mathbf{m}))} a_{\mathbf{m}}(v^2) u_{\mathbf{m}} \in \mathcal{H}(\Delta_n),$$

where the sum is taken over all $\mathbf{m} \in \mathfrak{M}_n$ such that $\dim M(\mathbf{m}) = t\delta$ with $\delta = (1, \dots, 1) \in \mathbb{N}I_n$, and $\text{soc } M(\mathbf{m})$ is square-free, i.e., $\dim \text{soc } M(\mathbf{m}) \leq \delta$. The following result is proved in [19].

Theorem 3.2. *The elements c_m are central in $\mathcal{H}(\Delta_n)$. Moreover, there is a decomposition*

$$\mathcal{H}(\Delta_n) = \mathcal{C}(\Delta_n) \otimes_{\mathbb{Q}(v)} \mathbb{Q}(v)[c_1, c_2, \dots],$$

where $\mathbb{Q}(v)[c_1, c_2, \dots]$ is the polynomial algebra in c_t for $t \geq 1$. In particular, $\mathcal{H}(\Delta_n)$ is generated by u_i and c_t for $i \in I_n$ and $t \geq 1$.

For each $\mathbf{m} \in \mathfrak{M}$, set $d(\mathbf{m}) = \dim M(\mathbf{m})$, $\mathbf{d}(\mathbf{m}) = \dim M(\mathbf{m})$ and define

$$(3.2.1) \quad \tilde{u}_{\mathbf{m}} = v^{\dim \text{End}_{k\Delta}(M(\mathbf{m})) - d(\mathbf{m})} u_{\mathbf{m}}.$$

Then $\{\tilde{u}_{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}\}$ is also a \mathcal{Z} -basis of $\mathcal{H}(\Delta)$ which plays a role in the construction of the canonical basis. In particular,

$$\tilde{u}_i = u_i \quad \text{for each } i \in I \quad \text{and} \quad \tilde{u}_{\mathbf{d}} = v^{\sum_i (d_i^2 - d_i)} u_{\mathbf{d}} \quad \text{for each } \mathbf{d} \in \mathbb{N}I.$$

Consider the map $\pi : \mathbb{Z}I_\infty \rightarrow \mathbb{Z}I_n$, $\mathbf{d} \mapsto \bar{\mathbf{d}}$, where $\pi(\mathbf{d}) = \bar{\mathbf{d}} = (d_{\bar{i}})$ is defined by

$$d_{\bar{i}} = \sum_{j \in \bar{i}} d_j, \quad \forall \bar{i} \in I_n = \mathbb{Z}/n\mathbb{Z}.$$

Then for each representation $M \in \text{Rep } \Delta_\infty$,

$$\dim \mathcal{F}(M) = \pi(\dim M).$$

Take $\mathbf{d} \in \mathbb{N}I_\infty$ with $\bar{\mathbf{d}} = \pi(\mathbf{d})$. By [45, 6.1], there is a \mathcal{Z} -linear map

$$\gamma_{\mathbf{d}} : \mathcal{H}(\Delta_n)_{\bar{\mathbf{d}}} \longrightarrow \mathcal{H}(\Delta_\infty)_{\mathbf{d}}.$$

The first two statements in the following lemma are taken from [45, Sect. 6.1], and the third one follows from the isomorphism $\tau : \mathcal{H}(\Delta_\infty) \rightarrow \mathcal{H}(\Delta_\infty)$.

Lemma 3.3. (1) *For each $\mathbf{d} \in \mathbb{N}I_\infty$, $\gamma_{\mathbf{d}}(\tilde{u}_{\bar{\mathbf{d}}}) = v^{-h(\mathbf{d})} \tilde{u}_{\mathbf{d}}$, where $h(\mathbf{d}) = \sum_{i < j, \bar{i} = \bar{j}} d_i(d_{j+1} - d_j)$.*

(2) *Fix $\alpha, \beta \in \mathbb{N}I_n$ with $\bar{\mathbf{d}} = \alpha + \beta$. Then for $x \in \mathcal{H}(\Delta_n)_\alpha$ and $y \in \mathcal{H}(\Delta_n)_\beta$,*

$$(3.3.1) \quad \sum_{\mathbf{a}, \mathbf{b}} v^{\kappa(\mathbf{a}, \mathbf{b})} \gamma_{\mathbf{a}}(x) \gamma_{\mathbf{b}}(y) = \gamma_{\mathbf{d}}(xy),$$

where the sum is taken over all pairs $\mathbf{a}, \mathbf{b} \in \mathbb{N}I_\infty$ satisfying $\mathbf{a} + \mathbf{b} = \mathbf{d}$, $\bar{\mathbf{a}} = \alpha$, and $\bar{\mathbf{b}} = \beta$, and $\kappa(\mathbf{a}, \mathbf{b}) = \sum_{i>j, \bar{i}=\bar{j}} a_i(2b_j - b_{j-1} - b_{j+1})$.

(3) For each $\mathbf{d} \in \mathbb{N}I_\infty$ and $\mathbf{m} \in \mathfrak{M}_n^{\mathbf{d}}$, $\gamma_{\tau^n(\mathbf{d})}(\tilde{u}_{\mathbf{m}}) = \tau^n(\gamma_{\mathbf{d}}(\tilde{u}_{\mathbf{m}}))$.

We now describe the images of the basis elements of $\mathcal{H}(\Delta_n)_{\bar{\mathbf{d}}}$ under $\gamma_{\mathbf{d}}$.

Proposition 3.4. *Let $\mathbf{d} \in \mathbb{N}I_\infty$ and $\mathbf{m} \in \mathfrak{M}_n$ be such that $\alpha := \dim M(\mathbf{m}) = \bar{\mathbf{d}}$. Then*

$$\gamma_{\mathbf{d}}(\tilde{u}_{\mathbf{m}}) \in \sum_{\mathfrak{z} \in \mathfrak{M}_\infty, \mathcal{F}(\mathfrak{z}) \leq_{\deg} \mathbf{m}} \mathcal{Z} \tilde{u}_{\mathfrak{z}}.$$

Proof. Consider the radical filtration of $M = M(\mathbf{m})$

$$M = \text{rad}^0 M \supseteq \text{rad} M \supseteq \cdots \supseteq \text{rad}^{\ell-1} M \supseteq \text{rad}^\ell M = 0$$

with $\text{rad}^{s-1} M / \text{rad}^s M \cong S_{\alpha_s}$, where ℓ is the Loewy length of M and $\alpha_s \in \mathbb{N}I_n$ for $1 \leq s \leq \ell$. Then $M = S_{\alpha_1} * \cdots * S_{\alpha_\ell}$. Moreover, by [8, Sect. 9],

$$\tilde{u}_{\alpha_1} \cdots \tilde{u}_{\alpha_\ell} = \tilde{u}_{\mathbf{m}} + \sum_{\mathbf{p} <_{\deg} \mathbf{m}} f_{\mathbf{m}, \mathbf{p}} \tilde{u}_{\mathbf{p}}, \text{ where } f_{\mathbf{m}, \mathbf{p}} \in \mathcal{Z}.$$

On the one hand, by induction with respect to the order \leq_{\deg} , we may assume that for each $\mathbf{p} \in \mathfrak{M}_n^{\mathbf{d}}$ with $\mathbf{p} <_{\deg} \mathbf{m}$, $\gamma_{\mathbf{d}}(\tilde{u}_{\mathbf{p}})$ is a \mathcal{Z} -linear combination of $\tilde{u}_{\mathfrak{z}}$ with $\mathfrak{z} \in \mathfrak{M}_\infty$ satisfying $\mathcal{F}(\mathfrak{z}) \leq_{\deg} \mathbf{p}$. Therefore,

$$\gamma_{\mathbf{d}}(\tilde{u}_{\mathbf{m}}) = \gamma_{\mathbf{d}}(\tilde{u}_{\alpha_1} \cdots \tilde{u}_{\alpha_\ell}) + x,$$

where $x = -\sum_{\mathbf{p} <_{\deg} \mathbf{m}} f_{\mathbf{m}, \mathbf{p}} \gamma_{\mathbf{d}}(\tilde{u}_{\mathbf{p}})$ is a \mathcal{Z} -linear combination of $\tilde{u}_{\mathfrak{z}}$ with $\mathcal{F}(\mathfrak{z}) <_{\deg} \mathbf{m}$.

On the other hand, by applying (3.3.1) inductively, we obtain

$$\gamma_{\mathbf{d}}(\tilde{u}_{\alpha_1} \cdots \tilde{u}_{\alpha_\ell}) = \sum_{\mathbf{a}_1, \dots, \mathbf{a}_\ell} v^{\sum_{s < t} \kappa(\mathbf{a}_s, \mathbf{a}_t) - \sum_s h(\mathbf{a}_s)} \tilde{u}_{\mathbf{a}_1} \cdots \tilde{u}_{\mathbf{a}_\ell},$$

where the sum is taken over all sequences $\mathbf{a}_1, \dots, \mathbf{a}_\ell \in \mathbb{N}I_\infty$ satisfying

$$\mathbf{a}_1 + \cdots + \mathbf{a}_\ell = \mathbf{d} \text{ and } \bar{\mathbf{a}}_s = \alpha_s, \forall 1 \leq s \leq \ell.$$

By the definition, each term $\tilde{u}_{\mathbf{a}_1} \cdots \tilde{u}_{\mathbf{a}_\ell}$ is a \mathcal{Z} -linear combination of $\tilde{u}_{\mathfrak{z}}$ such that $M(\mathfrak{z})$ admits a filtration

$$M(\mathfrak{z}) = X_0 \supset X_1 \supset \cdots \supset X_{\ell-1} \supset X_\ell = 0$$

such that $X_{s-1}/X_s \cong S_{\alpha_s}$ for all $1 \leq s \leq \ell$. Applying the exact functor \mathcal{F} gives a filtration of $\mathcal{F}(M(\mathfrak{z}))$

$$\mathcal{F}(M(\mathfrak{z})) = \mathcal{F}(X_0 \supset \mathcal{F}(X_1)) \supset \cdots \supset \mathcal{F}(X_{\ell-1}) \supset \mathcal{F}(X_\ell) = 0$$

such that

$$\mathcal{F}(X_{s-1})/\mathcal{F}(X_s) \cong \mathcal{F}(X_{s-1}/X_s) \cong S_{\alpha_s}, \forall 1 \leq s \leq \ell.$$

Therefore,

$$\mathcal{F}(M(\mathfrak{z})) = M(\mathcal{F}(\pi)) \leq_{\deg} S_{\alpha_1} * \cdots * S_{\alpha_\ell} = M(\mathbf{m}),$$

that is, $\mathcal{F}(\mathfrak{z}) \leq_{\deg} \mathbf{m}$.

In conclusion, we obtain that

$$\gamma_{\mathbf{d}}(\tilde{u}_{\mathbf{m}}) \in \sum_{\mathfrak{z} \in \mathfrak{M}_\infty, \mathcal{F}(\mathfrak{z}) \leq_{\deg} \mathbf{m}} \mathcal{Z} \tilde{u}_{\mathfrak{z}}.$$

□

Fix $\lambda \in \Pi$ and write

$$\mathbf{d}(\lambda) = \mathbf{dim} M(\mathbf{m}_\lambda^\infty) \in \mathbb{N}I_\infty \text{ and } \alpha(\lambda) = \mathbf{dim} M(\mathbf{m}_\lambda) \in \mathbb{N}I_n.$$

By the definition of $M(\mathbf{m}_\lambda^\infty)$ and $M(\mathbf{m}_\lambda)$, the radical filtration of $\widetilde{M} = M(\mathbf{m}_\lambda^\infty)$

$$\widetilde{M} = \text{rad}^0 \widetilde{M} \supseteq \text{rad} \widetilde{M} \supseteq \cdots \supseteq \text{rad}^{\ell-1} \widetilde{M} \supseteq \text{rad}^\ell \widetilde{M} = 0$$

gives rise to the radical filtration of $M(\mathbf{m}_\lambda) = \mathcal{F}(\widetilde{M})$

$$M(\mathbf{m}_\lambda) = \mathcal{F}(\text{rad}^0 \widetilde{M}) \supseteq \mathcal{F}(\text{rad} \widetilde{M}) \supseteq \cdots \supseteq \mathcal{F}(\text{rad}^{\ell-1} \widetilde{M}) \supseteq \mathcal{F}(\text{rad}^\ell \widetilde{M}) = 0,$$

that is, $\mathcal{F}(\text{rad}^s \widetilde{M}) = \text{rad}^s(M(\mathbf{m}_\lambda))$ for $1 \leq s \leq \ell$. Let $\mathbf{d}(\lambda)_s \in \mathbb{N}I_\infty$ and $\alpha(\lambda)_s \in \mathbb{N}I_n$, $1 \leq s \leq \ell$, be such that

$$\text{rad}^{s-1} \widetilde{M} = \text{rad}^s \widetilde{M} \cong S_{\mathbf{d}(\lambda)_s} \text{ and } \text{rad}^{s-1} M(\mathbf{m}_\lambda) / \text{rad}^s M(\mathbf{m}_\lambda) \cong S_{\alpha(\lambda)_s}.$$

Then $\overline{\mathbf{d}(\lambda)}_s = \alpha(\lambda)_s$ for $1 \leq s \leq \ell$. Applying the above proposition to \mathbf{m}_λ gives the following result.

Corollary 3.5. (1) *Let $\lambda \in \Pi$ and keep the notation above. Then*

$$\gamma_{\mathbf{d}(\lambda)}(\widetilde{u}_{\mathbf{m}_\lambda}) \in v^{\theta(\lambda)} \widetilde{u}_{\mathbf{m}_\lambda^\infty} + \sum_{\mathfrak{z} \in \mathfrak{M}_\infty, \mathcal{F}(\mathfrak{z}) < \deg \mathbf{m}_\lambda} \mathcal{Z} \widetilde{u}_{\mathfrak{z}},$$

where $\theta(\lambda) = \sum_{s < t} \kappa(\mathbf{d}(\lambda)_s, \mathbf{d}(\lambda)_t) - \sum_{s=1}^\ell h(\mathbf{d}(\lambda)_s)$.

(2) *Let $\mathbf{d} \in \mathbb{N}I_\infty$ with $\overline{\mathbf{d}} = \alpha(\lambda)$. If $\mathbf{d} = \tau^{rn}(\mathbf{d}(\lambda))$ for some $r \in \mathbb{Z}$, then*

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathbf{m}_\lambda}) \in v^{\theta(\lambda)} \widetilde{u}_{\tau^{rn}(\mathbf{m}_\lambda^\infty)} + \sum_{\mathfrak{z} \in \mathfrak{M}_\infty, \mathcal{F}(\mathfrak{z}) < \deg \mathbf{m}_\lambda} \mathcal{Z} \widetilde{u}_{\mathfrak{z}}.$$

Otherwise,

$$\gamma_{\mathbf{d}}(\widetilde{u}_{\mathbf{m}_\lambda}) \in \sum_{\mathfrak{z} \in \mathfrak{M}_\infty^{\mathbf{d}}, \mathcal{F}(\mathfrak{z}) < \deg \mathbf{m}_\lambda} \mathcal{Z} \widetilde{u}_{\mathfrak{z}}.$$

In the following we briefly recall the canonical basis of $\mathcal{H}(\Delta)$ for $\Delta = \Delta_n$ or Δ_∞ . By [31] and [45, Prop. 7.5], there is a semilinear ring involution $\iota : \mathcal{H}(\Delta) \rightarrow \mathcal{H}(\Delta)$ taking $v \mapsto v^{-1}$ and $\widetilde{u}_{\mathbf{d}} \mapsto \widetilde{u}_{\mathbf{d}}$ for all $\mathbf{d} \in \mathbb{Z}I_n$. It is often called the bar-involution, usually written as $\bar{x} = \iota(x)$. The canonical basis (or the global crystal basis in the sense of Kashiwara) $\mathbf{B} := \{b_{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}\}$ for $\mathcal{H}(\Delta)$ (at $v = \infty$) can be characterized as follows:

$$(3.5.1) \quad \bar{b}_{\mathbf{m}} = b_{\mathbf{m}}, \quad b_{\mathbf{m}} \in \widetilde{u}_{\mathbf{m}} + \sum_{\mathfrak{p} < \deg \mathbf{m}} v^{-1} \mathbb{Z}[v^{-1}] \widetilde{u}_{\mathfrak{p}};$$

see [31]. The canonical basis elements $b_{\mathbf{m}}$ also admit a geometric characterization given in [32, 45]. Let $H_{\mathcal{O}_{\mathfrak{p}}}^i(IC_{\mathcal{O}_{\mathbf{m}}})$ be the stalk at a point of $\mathcal{O}_{\mathbf{m}}$ of the i -th intersection cohomology sheaf of the closure $\overline{\mathcal{O}_{\mathfrak{p}}}$ of $\mathcal{O}_{\mathfrak{p}}$. Then

$$b_{\mathbf{m}} = \sum_{\substack{i \in \mathbb{N} \\ \mathfrak{p} \leq \deg \mathbf{m}}} v^{i - \dim \mathcal{O}_{\mathbf{m}} + \dim \mathcal{O}_{\mathfrak{p}}} \dim H_{\mathcal{O}_{\mathfrak{p}}}^i(IC_{\mathcal{O}_{\mathbf{m}}}) \widetilde{u}_{\mathfrak{p}}.$$

For the cyclic quiver case, by [33], the subset of \mathbf{B}

$$\mathbf{B}^{\text{ap}} := \{b_{\mathbf{m}} \mid \mathbf{m} \in \mathfrak{M}_n^{\text{ap}}\}$$

is the canonical basis of $\mathcal{C}(\Delta_n)$, where $\mathfrak{M}_n^{\text{ap}}$ denotes the set of aperiodic multisegments, that is, those multisegments $\mathbf{m} = \sum_{i \in I_n, l \geq 1} m_{i,l}[i, l]$ satisfying that for each $l \geq 1$, there is some $i \in I_n$ such that $m_{i,l} = 0$. In other words, \mathbf{B}^{ap} is the canonical basis of $\mathbf{U}_v^\pm(\widehat{\mathfrak{sl}}_n)$. Note that for each $\lambda = (\lambda_1, \dots, \lambda_m) \in \Pi$, the corresponding multisegment \mathbf{m}_λ is aperiodic if and only if λ is n -regular which, by definition, satisfies $\lambda_s > \lambda_{s+n-1}$ for $1 \leq s \leq s+n-1 \leq m$.

4. DOUBLE RINGEL–HALL ALGEBRAS AND HIGHEST WEIGHT MODULES

In this section we follow [44, 6] to define the double Ringel–Hall algebra $\mathcal{D}(\Delta)$ of the quiver $\Delta = \Delta_n$ or Δ_∞ and study the irreducible highest weight modules of $\mathcal{D}(\Delta_n)$ associated with integral dominant weights in terms of a quantized generalized Kac–Moody algebra.

The Ringel–Hall algebra $\mathcal{H}(\Delta)$ of Δ can be extended to a Hopf algebra $\mathcal{D}(\Delta)^{\geq 0}$ which is a $\mathbb{Q}(v)$ -vector space with a basis $\{u_{\mathfrak{m}}^+ K_\alpha \mid \alpha \in \mathbb{Z}I, \mathfrak{m} \in \mathfrak{M}\}$; see [38, 15, 44] or [6, Prop. 1.5.3]. Its algebra structure is given by

$$(4.0.2) \quad \begin{aligned} K_\alpha K_\beta &= K_{\alpha+\beta}, \quad K_\alpha u_{\mathfrak{m}}^+ = v^{\langle \mathbf{d}(\mathfrak{m}), \alpha \rangle} u_{\mathfrak{m}}^+ K_\alpha, \\ u_{\mathfrak{m}}^+ u_{\mathfrak{m}'}^+ &= \sum_{\mathfrak{p} \in \mathfrak{M}} v^{\langle \mathbf{d}(\mathfrak{m}), \mathbf{d}(\mathfrak{m}') \rangle} \varphi_{\mathfrak{m}, \mathfrak{m}'}^{\mathfrak{p}}(v^2) u_{\mathfrak{p}}^+, \end{aligned}$$

where $\mathfrak{m}, \mathfrak{m}' \in \mathfrak{M}$ and $\alpha, \beta \in \mathbb{Z}I$, and its coalgebra structure is given by

$$(4.0.3) \quad \begin{aligned} \Delta(u_{\mathfrak{m}}^+) &= \sum_{\mathfrak{m}', \mathfrak{m}'' \in \mathfrak{M}} v^{\langle \mathbf{d}(\mathfrak{m}'), \mathbf{d}(\mathfrak{m}'') \rangle} \frac{\mathfrak{a}_{\mathfrak{m}'}(v^2) \mathfrak{a}_{\mathfrak{m}''}(v^2)}{\mathfrak{a}_{\mathfrak{m}}(v^2)} \varphi_{\mathfrak{m}', \mathfrak{m}''}^{\mathfrak{m}}(v^2) u_{\mathfrak{m}'}^+ \otimes u_{\mathfrak{m}''}^+ K_{\mathbf{d}(\mathfrak{m}')}, \\ \Delta(K_\alpha) &= K_\alpha \otimes K_\alpha, \quad \varepsilon(u_{\mathfrak{m}}^+) = 0 \ (\mathfrak{m} \neq 0), \quad \varepsilon(K_\alpha) = 1, \end{aligned}$$

where $\mathfrak{m} \in \mathfrak{M}$ and $\alpha \in \mathbb{Z}I$. We refer to [44] or [6] for the definition of the antipode.

Dually, there is a Hopf algebra $\mathcal{D}(\Delta)^{\leq 0}$ with basis $\{K_\alpha u_{\mathfrak{m}}^- \mid \alpha \in \mathbb{Z}I, \mathfrak{m} \in \mathfrak{M}\}$. In particular, the multiplication is given by

$$(4.0.4) \quad \begin{aligned} K_\alpha K_\beta &= K_{\alpha+\beta}, \quad K_\alpha u_{\mathfrak{m}}^- = v^{-\langle \mathbf{d}(\mathfrak{m}), \alpha \rangle} u_{\mathfrak{m}}^- K_\alpha, \\ u_{\mathfrak{m}}^- u_{\mathfrak{m}'}^- &= \sum_{\mathfrak{p} \in \mathfrak{M}} v^{\langle \mathbf{d}(\mathfrak{m}'), \mathbf{d}(\mathfrak{m}) \rangle} \varphi_{\mathfrak{m}', \mathfrak{m}}^{\mathfrak{p}}(v^2) u_{\mathfrak{p}}^-, \end{aligned}$$

where $\mathfrak{m}, \mathfrak{m}' \in \mathfrak{M}$ and $\alpha, \beta \in \mathbb{Z}I$. The comultiplication and the counit are given by

$$(4.0.5) \quad \begin{aligned} \Delta(u_{\mathfrak{m}}^-) &= \sum_{\mathfrak{m}', \mathfrak{m}'' \in \mathfrak{M}} v^{\langle \mathbf{d}(\mathfrak{m}'), \mathbf{d}(\mathfrak{m}'') \rangle} \frac{\mathfrak{a}_{\mathfrak{m}'} \mathfrak{a}_{\mathfrak{m}''}}{\mathfrak{a}_{\mathfrak{m}}} \varphi_{\mathfrak{m}', \mathfrak{m}''}^{\mathfrak{m}}(v^2) u_{\mathfrak{m}'}^- K_{-\mathbf{d}(\mathfrak{m}')} \otimes u_{\mathfrak{m}''}^-, \\ \Delta(K_\alpha) &= K_\alpha \otimes K_\alpha, \quad \varepsilon(u_{\mathfrak{m}}^-) = 0 \ (\mathfrak{m} \neq 0), \quad \varepsilon(K_\alpha) = 1, \end{aligned}$$

where $\alpha \in \mathbb{Z}I$ and $\mathfrak{m} \in \mathfrak{M}$.

It is routine to check that the bilinear form $\psi : \mathcal{D}(\Delta)^{\geq 0} \times \mathcal{D}(\Delta)^{\leq 0} \rightarrow \mathbb{Q}(v)$ defined by

$$(4.0.6) \quad \psi(K_\alpha u_{\mathfrak{m}}^+, K_\beta u_{\mathfrak{m}'}^-) = v^{(\alpha, \beta) - \langle \mathbf{d}(\mathfrak{m}), \mathbf{d}(\mathfrak{m}') \rangle + 2d(\mathfrak{m})} \frac{\delta_{\mathfrak{m}, \mathfrak{m}'}}{\mathfrak{a}_{\mathfrak{m}}(v^2)}$$

is a skew-Hopf pairing in the sense of [24]; see, for example, [6, Prop. 2.1.3].

Following [44] or [6, §2.1], with the triple $(\mathcal{D}(\Delta)^{\geq 0}, \mathcal{D}(\Delta)^{\leq 0}, \psi)$ we obtain the associated reduced double Ringel–Hall algebra $\mathcal{D}(\Delta)$ which inherits a Hopf algebra structure from those of $\mathcal{D}(\Delta)^{\geq 0}$ and $\mathcal{D}(\Delta)^{\leq 0}$. In particular, for all elements $x \in \mathcal{D}(\Delta)^{\geq 0}$ and $y \in \mathcal{D}(\Delta)^{\leq 0}$, we have in $\mathcal{D}(\Delta)^{\leq 0}$ the following relations

$$(4.0.7) \quad \sum \psi(x_1, y_1) y_2 x_2 = \sum \psi(x_2, y_2) x_1 y_1,$$

where $\Delta(x) = \sum x_1 \otimes x_2$ and $\Delta(y) = \sum y_1 \otimes y_2$. Moreover, $\mathcal{D}(\Delta)$ admits a triangular decomposition

$$(4.0.8) \quad \mathcal{D}(\Delta) = \mathcal{D}(\Delta)^+ \otimes \mathcal{D}(\Delta)^0 \otimes \mathcal{D}(\Delta)^-,$$

where $\mathcal{D}(\Delta)^\pm$ are subalgebras generated by u_m^\pm ($m \in \mathfrak{M}$), and $\mathcal{D}(\Delta)^0$ is generated by K_α ($\alpha \in \mathbb{Z}I$). Thus, $\mathcal{D}(\Delta)^0$ is identified with the Laurent polynomial ring $\mathbb{Q}(v)[K_i^{\pm 1} : i \in I]$,

$$\begin{aligned}\mathcal{H}(\Delta) &= \mathcal{H}(\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}(v) \xrightarrow{\sim} \mathcal{D}(\Delta)^+, \quad u_m \mapsto u_m^+, \\ \mathcal{H}(\Delta)^{\text{op}} &= \mathcal{H}(\Delta)^{\text{op}} \otimes_{\mathbb{Z}} \mathbb{Q}(v) \xrightarrow{\sim} \mathcal{D}(\Delta)^-, \quad u_m \mapsto u_m^-. \end{aligned}$$

The canonical basis of $\mathcal{H}(\Delta)$ given in (3.5.1) gives the canonical bases $\mathbf{B}^\pm := \{b_m^\pm \mid m \in \mathfrak{M}\}$ of $\mathcal{D}(\Delta)^\pm$ satisfying

$$(4.0.9) \quad b_m^\pm \in \tilde{u}_m^\pm + \sum_{p < \deg m} v^{-1} \mathbb{Z}[v^{-1}] \tilde{u}_p^\pm.$$

For $i \in I$, $\alpha \in \mathbb{N}I$ and $m \in \mathfrak{M}$, we write

$$u_i^\pm = u_{[S_i]}^\pm, \quad u_\alpha^\pm = u_{[S_\alpha]}^\pm, \quad \text{and} \quad \tilde{u}_m^\pm = v^{\dim \text{End}_\Delta(M(m)) - \dim M(m)} u_m^\pm.$$

It is known that $\mathcal{D}(\Delta_\infty)$ is generated by $u_i^\pm, K_i^{\pm 1}$ ($i \in \mathbb{Z}$) and is isomorphic to $\mathbf{U}_v(\mathfrak{sl}_\infty)$. By [39], the $\mathbb{Q}(v)$ -subalgebra of $\mathcal{D}(\Delta_n)$ generated by $u_i^\pm, K_i^{\pm 1}$ ($i \in I_n = \mathbb{Z}/n\mathbb{Z}$) is isomorphic to $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$, while $\mathcal{D}(\Delta_n)$ is isomorphic to $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$; see [41, 21, 6]. From now on, we write for notational simplicity,

$$\mathcal{D}(\infty) = \mathcal{D}(\Delta_\infty) \quad \text{and} \quad \mathcal{D}(n) = \mathcal{D}(\Delta_n).$$

Remarks 4.1. (1) The construction of $\mathcal{D}(n)$ is slightly different from that in [6, §2.1]. In particular, the K_i here play a role as $\tilde{K}_i = K_i K_{i+1}^{-1}$ there. In particular, they do not satisfy the equality $K_0 K_1 \cdots K_{n-1} = 1$.

(2) We can extend $\mathcal{D}(n)$ to the $\mathbb{Q}(v)$ -algebra $\widehat{\mathcal{D}}(n)$ by adding new generators $D^{\pm 1}$ with relations

$$DD^{-1} = 1 = D^{-1}D, \quad K_i D = D K_i, \quad D E_i = v^{\delta_{0,i}} E_i D, \quad D u_m^\pm = v^{-a_0} u_m^\pm D$$

for all $i \in I_n$ and $m \in \mathfrak{M}$, where $\mathbf{d}(m) = (a_i)_{i \in I_n}$. Then $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ clearly becomes a subalgebra of $\widehat{\mathcal{D}}(n)$.

As in (3.1.1), define for each $t \geq 1$,

$$\mathbf{c}_t^\pm = (-1)^t v^{-2tn} \sum_m (-1)^{\dim \text{End}(M(m))} \mathbf{a}_m(v^2) u_m^\pm \in \mathcal{D}(n)^\pm,$$

By Theorem 3.2, the elements \mathbf{c}_t^+ and \mathbf{c}_t^- are central in $\mathcal{D}(n)^+$ and $\mathcal{D}(n)^-$, respectively. Following [21, Sect. 4], define recursively for $t \geq 1$,

$$\mathbf{x}_t^\pm = t \mathbf{c}_t^\pm - \sum_{s=1}^{t-1} \mathbf{x}_s^\pm \mathbf{c}_{t-s}^\pm \in \mathcal{D}(n)^\pm.$$

Clearly, \mathbf{x}_t^+ and \mathbf{x}_t^- are again central elements in $\mathcal{D}(n)^+$ and $\mathcal{D}(n)^-$, respectively. By applying [19, Cor. 10 & 12], the \mathbf{x}_t^\pm are primitive, i.e.,

$$\Delta(\mathbf{x}_t^+) = \mathbf{x}_t^+ \otimes K_{t\delta} + 1 \otimes \mathbf{x}_t^+ \quad \text{and} \quad \Delta(\mathbf{x}_t^-) = \mathbf{x}_t^- \otimes 1 + K_{-t\delta} \otimes \mathbf{x}_t^-,$$

and they satisfy

$$\psi(\mathbf{x}_t^+, \mathbf{x}_s^-) = v^{2tn} \{\mathbf{x}_t, \mathbf{x}_s\} = \delta_{t,s} t v^{2tn} v^{-2tn} (1 - v^{-2tn}) = \delta_{t,s} t (1 - v^{-2tn}).$$

Finally, as in [6, § 2.2], we scale the elements \mathbf{x}_t^\pm by setting

$$\mathbf{z}_t^\pm = \frac{v^{tn}}{v^t - v^{-t}} \mathbf{x}_t^\pm \in \mathcal{D}(n)^\pm \quad \text{for } t \geq 1.$$

Then

$$(4.1.1) \quad \Delta(\mathbf{z}_t^+) = \mathbf{z}_t^+ \otimes K_{t\delta} + 1 \otimes \mathbf{z}_t^+, \quad \Delta(\mathbf{z}_t^-) = \mathbf{z}_t^- \otimes 1 + K_{-t\delta} \otimes \mathbf{z}_t^-,$$

and

$$\psi(\mathbf{z}_t^+, \mathbf{z}_s^-) = \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2}.$$

Lemma 4.2. (1) For each $i \in I_n$,

$$[u_i^+, u_i^-] = \frac{K_i - K_i^{-1}}{v - v^{-1}}.$$

(2) For $\alpha \in \mathbb{N}I_n$ and $t, s \geq 1$, $K_\alpha \mathbf{z}_t^\pm = \mathbf{z}_t^\pm K_\alpha$ and

$$(4.2.1) \quad [\mathbf{z}_t^+, \mathbf{z}_s^-] = \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2} (K_{t\delta} - K_{-t\delta}).$$

Moreover, for each $i \in I_n$ and $t \geq 1$,

$$[u_i^+, \mathbf{z}_t^-] = 0 = [u_i^-, \mathbf{z}_t^+].$$

Proof. We only prove the formula (4.2.1). The remaining ones are obvious. Since $\Delta(\mathbf{z}_t^+) = \mathbf{z}_t^+ \otimes K_{t\delta} + 1 \otimes \mathbf{z}_t^+$ and $\Delta(\mathbf{z}_s^-) = \mathbf{z}_s^- \otimes 1 + K_{-s\delta} \otimes \mathbf{z}_s^-$, we have by (4.0.7) that

$$\begin{aligned} & K_{t\delta} \psi(\mathbf{z}_t^+, \mathbf{z}_s^-) + \mathbf{z}_t^+ \psi(1, \mathbf{z}_s^-) + \mathbf{z}_s^- K_{t\delta} \psi(\mathbf{z}_t^+, K_{-s\delta}) + \mathbf{z}_s^- \mathbf{z}_t^+ \psi(1, K_{-s\delta}) \\ &= \mathbf{z}_t^+ \mathbf{z}_s^- \psi(K_{t\delta}, 1) + \mathbf{z}_s^- \psi(\mathbf{z}_t^+, 1) + \mathbf{z}_t^+ K_{-s\delta} \psi(K_{t\delta}, \mathbf{z}_s^-) + K_{-s\delta} \psi(\mathbf{z}_t^+, \mathbf{z}_s^-). \end{aligned}$$

This implies that

$$[\mathbf{z}_t^+, \mathbf{z}_s^-] = \psi(\mathbf{z}_t^+, \mathbf{z}_s^-) (K_{t\delta} - K_{-s\delta}) = \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2} (K_{t\delta} - K_{-t\delta})$$

since $\psi(1, \mathbf{z}_s^-) = \psi(\mathbf{z}_t^+, K_{s\delta}) = \psi(\mathbf{z}_t^+, 1) = \psi(K_{t\delta}, \mathbf{z}_s^-) = 0$ and $\psi(1, K_{s\delta}) = \psi(K_{-t\delta}, 1) = 1$. \square

Using arguments similar to those in the proof of [6, Th. 2.3.1], we obtain a presentation of $\mathcal{D}(n)$. More precisely, $\mathcal{D}(n)$ is the $\mathbb{Q}(v)$ -algebra generated by $K_i^{\pm 1}$, $u_i^+ = E_i$, $u_i^- = F_i$, and \mathbf{z}_t^\pm for $i \in I_n$ and $t \geq 1$ with defining relations:

$$\begin{aligned} \text{(DH1)} \quad & K_i K_j = K_j K_i, \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i; \\ \text{(DH2)} \quad & K_i E_j = v^{a_{ij}} E_j K_i, \quad K_i F_j = v^{-a_{ij}} F_j K_i, \quad K_i \mathbf{z}_t^\pm = \mathbf{z}_t^\pm K_i; \\ \text{(DH3)} \quad & [E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{v - v^{-1}}, \quad [E_i, \mathbf{z}_t^-] = 0, \quad [\mathbf{z}_t^+, F_i] = 0, \\ & [\mathbf{z}_t^+, \mathbf{z}_s^-] = \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2} (K_{t\delta} - K_{-t\delta}); \\ \text{(DH4)} \quad & \sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1 - c_{i,j} \\ a \end{bmatrix} E_i^a E_j E_i^b = 0 \text{ for } i \neq j, \\ & \mathbf{z}_t^+ \mathbf{z}_s^+ = \mathbf{z}_s^+ \mathbf{z}_t^+, \quad E_i \mathbf{z}_t^+ = \mathbf{z}_t^+ E_i; \\ \text{(DH5)} \quad & \sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1 - c_{i,j} \\ a \end{bmatrix} F_i^a F_j F_i^b = 0 \text{ for } i \neq j, \\ & \mathbf{z}_t^- \mathbf{z}_s^- = \mathbf{z}_s^- \mathbf{z}_t^-, \quad F_i \mathbf{z}_t^- = \mathbf{z}_t^- F_i, \end{aligned}$$

where $i, j \in I_n$ and $t, s \geq 1$.

In the following we simply identify $I_n = \mathbb{Z}/n\mathbb{Z}$ with the subset $\{0, 1, \dots, n-1\}$ of \mathbb{Z} . Let $P^\vee = (\oplus_{i \in I_n} \mathbb{Z} h_i) \oplus \mathbb{Z} d$ be the free abelian group with basis $\{h_i \mid i \in I_n\} \cup \{d\}$. Set $\mathfrak{h} = P^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ and define

$$P = \{\Lambda \in \mathfrak{h}^* = \text{Hom}_{\mathbb{Q}}(\mathfrak{h}, \mathbb{Q}) \mid \Lambda(P^\vee) \subset \mathbb{Z}\}.$$

Then $P = (\oplus_{i \in I_n} \mathbb{Z} \Lambda_i) \oplus \mathbb{Z} \omega$, where $\{\Lambda_i \mid i \in I_n\} \cup \{\omega\}$ is the dual basis of $\{h_i \mid i \in I_n\} \cup \{d\}$. This gives rise to the Cartan datum $(P^\vee, P, \Pi^\vee, \Pi)$ associated with the Cartan matrix $C_n = (a_{ij})$, where

$\Pi^\vee = \{h_i \mid i \in I_n\}$ is set of simple coroots and $\Pi = \{\alpha_i \mid i \in I_n\}$ is the set of simple roots defined by

$$\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d) = \delta_{0,i} \quad \text{for all } i, j \in I_n.$$

Finally, let

$$P^+ = \{\Lambda \in P \mid \Lambda(h_i) \geq 0, \forall i \in I_n\} = \left(\bigoplus_{i \in I_n} \mathbb{N}\Lambda_i \right) \oplus \mathbb{Z}\omega$$

denote the set of dominant weights.

For each $\Lambda \in X$, consider the left ideal J_Λ of $\mathcal{D}(n)$ defined by

$$\begin{aligned} J_\Lambda &= \sum_{\mathfrak{m} \in \mathfrak{M}_n \setminus \{0\}} \mathcal{D}(n)u_{\mathfrak{m}}^+ + \sum_{\alpha \in \mathbb{Z}I_n} \mathcal{D}(n)(K_\alpha - v^{\Lambda(\alpha)}) \\ &= \sum_{\mathfrak{m} \in \mathfrak{M}_n \setminus \{0\}} \mathcal{D}(n)u_{\mathfrak{m}}^+ + \sum_{i \in I_n} \mathcal{D}(n)(K_i - v^{\Lambda(h_i)}), \end{aligned}$$

where $\Lambda(\alpha) = \sum_{i \in I_n} a_i \Lambda(h_i)$ if $\alpha = \sum_{i \in I_n} a_i \varepsilon_i \in \mathbb{Z}I_n$. The quotient module

$$M(\Lambda) := \mathcal{D}(n)/J_\Lambda$$

is called the Verma module which is a highest weight module with highest vector $\eta_\Lambda := 1 + J_\Lambda$. Applying the triangular decomposition (4.0.8) shows that

$$\mathcal{D}(n)^- \longrightarrow M(\Lambda), \quad x^- \longmapsto x^- + J_\Lambda$$

is an isomorphism of $\mathbb{Q}(v)$ -vector spaces. Via this isomorphism, $\mathcal{D}(n)^-$ becomes a $\mathcal{D}(n)$ -module. It is clear that $M(\Lambda)$ contains a unique maximal submodule M' . This gives an irreducible $\mathcal{D}(n)$ -module $L(\Lambda) = M(\Lambda)/M'$.

Remark 4.3. By the construction, if $\Lambda, \Lambda' \in P^+$ satisfy $\Lambda - \Lambda' \in \mathbb{Z}\omega$, then $L(\Lambda) = L(\Lambda')$. Therefore, it might be more appropriate to work with the algebra $\widehat{\mathcal{D}}(n)$ defined in Remark 4.1(2).

Proposition 4.4. Let $\Lambda = \sum_{i \in I_n} a_i \Lambda_i + b\omega \in P^+$ be a dominant weight with $\sum_{i \in I_n} a_i > 0$. Then

$$L(\Lambda) \cong \mathcal{D}(n)^- / \left(\sum_{i \in I_n} \mathcal{D}(n)^- (u_i^-)^{a_i+1} \right).$$

Proof. As in [8, Sect. 3], we extend the Cartan matrix $C = (a_{ij})_{i,j \in I_n}$ to a Borcherds–Cartan matrix $\tilde{C} = (\tilde{a}_{ij})_{i,j \in \mathbb{N}}$ by setting $\tilde{a}_{ij} = a_{ij}$ for $0 \leq i, j < n$ and $\tilde{a}_{ij} = 0$ otherwise. Consider the free abelian group $\tilde{P}^\vee = (\oplus_{i \in \mathbb{N}} \mathbb{Z}h_i) \oplus (\oplus_{i \in \mathbb{N}} \mathbb{Z}d_i)$ and define

$$\tilde{P} = \{\theta \in (\tilde{P}^\vee \otimes \mathbb{Q})^* \mid \theta(\tilde{P}^\vee) \subset \mathbb{Z}\}.$$

We then obtain a Cartan datum of type \tilde{C}

$$(\tilde{P}^\vee, \tilde{P}, \tilde{\Pi}^\vee = \{h_i \mid i \in \mathbb{N}\}, \tilde{\Pi} = \{\tilde{\alpha}_i \mid i \in \mathbb{N}\})$$

where the $\tilde{\alpha}_i$ are defined by

$$\tilde{\alpha}_i(h_j) = \tilde{a}_{ji} \quad \text{and} \quad \tilde{\alpha}_i(d_j) = \delta_{i,j}, \quad \forall i, j \in \mathbb{N}.$$

Following [25, Def. 2.1] or [23, Def. 1.3], with the above Cartan datum we have the associated quantum generalized Kac–Moody algebra $\mathbf{U}_v(\tilde{C})$ which is by definition a $\mathbb{Q}(v)$ -algebra generated by $K_i^{\pm 1}, D_i^{\pm 1}, E_i, F_i$ for $i \in \mathbb{N}$ with relations; see [23, (1.4)] for the details. Clearly, the subalgebra of $\mathbf{U}_v(\tilde{C})$ generated by $K_i^{\pm 1}, D_0^{\pm 1}, E_i, F_i$ for $0 \leq i < n$ is isomorphic to $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$.

In order to make a comparison with $\mathcal{D}(n)$, we consider the subalgebra $\tilde{\mathbf{U}}$ of $\mathbf{U}_v(\tilde{C})$ generated by $K_i^{\pm 1}, E_i, F_i$ for $i \in \mathbb{N}$. Then $\tilde{\mathbf{U}}$ admits a triangular decomposition

$$\tilde{\mathbf{U}} = \tilde{\mathbf{U}}^- \otimes \tilde{\mathbf{U}}^0 \otimes \tilde{\mathbf{U}}^+,$$

where $\tilde{\mathbf{U}}^-$, $\tilde{\mathbf{U}}^+$, and $\tilde{\mathbf{U}}^0$ are subalgebras generated by F_i , E_i , and $K_i^{\pm 1}$ for $i \in \mathbb{N}$, respectively. In particular, $\tilde{\mathbf{U}}^0 = \mathbb{Q}(v)[K_i^{\pm 1} : i \in \mathbb{N}]$. It follows from the definition that there is a surjective algebra homomorphism $\Psi : \tilde{\mathbf{U}} \rightarrow \mathcal{D}(n)$ given by

$$\Psi(E_i) = \begin{cases} u_i^+, & \text{if } 0 \leq i < n; \\ y_{i-n+1} z_{i-n+1}^+, & \text{if } i \geq n, \end{cases} \quad \Psi(F_i) = \begin{cases} u_i^-, & \text{if } 0 \leq i < n; \\ z_{i-n+1}^-, & \text{if } i \geq n \end{cases}, \quad \text{and}$$

$$\Psi(K_i^{\pm 1}) = \begin{cases} K_i^{\pm 1}, & \text{if } 0 \leq i < n; \\ K_{(i-n+1)\delta}^{\pm 1}, & \text{if } i \geq n, \end{cases}$$

where $y_t = t(v^{2tn} - 1)(v - v^{-1})/(v^t - v^{-t})^2$ for $t \geq 1$; see (4.2.1). Hence, each $\mathcal{D}(n)$ -module can be viewed as a $\tilde{\mathbf{U}}$ -module via the homomorphism Ψ . In what follows, we will identify $\tilde{\mathbf{U}}^\pm$ with $\mathcal{D}(n)^\pm$ via Ψ .

As defined in [23, Sect. 2.1], for each $\theta \in \tilde{P}$, there is an associated irreducible $\tilde{\mathbf{U}}$ -module $L(\theta)$. By [23, Prop. 3.3], $L(\theta)$ is integrable if and only if θ is dominant, that is,

$$\theta \in \tilde{P}^+ = \{\rho \in (\tilde{P}^\vee \otimes \mathbb{Q})^* \mid \rho(\tilde{P}^\vee) \subset \mathbb{N}\}.$$

Moreover, by [25, Cor. 4.7],

$$L(\theta) \cong \tilde{\mathbf{U}}^- / \left(\sum_{i \in I_n} \tilde{\mathbf{U}}^- F_i^{\theta(h_i)+1} + \sum_{i \geq n, \theta(h_i)=0} \tilde{\mathbf{U}}^- F_i \right).$$

Viewing the irreducible $\mathcal{D}(n)$ -module $L(\Lambda)$ as a $\tilde{\mathbf{U}}$ -module, it is then isomorphic to $L(\tilde{\Lambda})$, where $\tilde{\Lambda} \in \tilde{P}$ is defined by

$$\tilde{\Lambda}(h_i) = \begin{cases} \Lambda(h_i) = a_i, & \text{if } 0 \leq i < n; \\ (i - n + 1) \sum_{0 \leq j < n} a_j, & \text{if } i \geq n \end{cases} \quad \text{and} \quad \tilde{\Lambda}(d_i) = \delta_{i,0} b.$$

From the assumption $\sum_{i \in I} a_i > 0$ it follows that $\tilde{\Lambda}(h_i) > 0$ for all $i \geq n$. Consequently,

$$L(\Lambda) \cong L(\tilde{\Lambda}) \cong \tilde{\mathbf{U}}^- / \left(\sum_{i \in I_n} \tilde{\mathbf{U}}^- F_i^{a_i+1} \right) = \mathcal{D}(n)^- / \left(\sum_{i \in I_n} \mathcal{D}(n)^- (u_i^-)^{a_i+1} \right).$$

□

For each $\Lambda \in P$, let $L_0(\Lambda)$ denote the irreducible $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$ -module of highest weight Λ . Applying Theorem 3.2 gives the following result.

Corollary 4.5. *Let $\Lambda = \sum_{i \in I_n} a_i \Lambda_i + b\omega \in P^+$ with $\sum_{i \in I_n} a_i > 0$. Then $L_0(\Lambda)$ is the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$ -submodule of $L(\Lambda)$ generated by the highest weight vector η_Λ and there is a vector space decomposition*

$$L(\Lambda) = L_0(\Lambda) \otimes \mathbb{Q}(v)[z_1^-, z_2^-, \dots].$$

In particular, if $L(\Lambda)|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)}$ denotes the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$ -module via restriction, then

$$(4.5.1) \quad L(\Lambda)|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)} \cong \bigoplus_{m \geq 0} L_0(\Lambda - m\delta^*)^{\oplus p(m)},$$

where $\delta^* = \sum_{i \in I_n} \alpha_i$ and $p(m)$ is the number of partitions of m .

Proof. By Theorem 3.2,

$$\mathcal{D}(n)^- = \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) \otimes \mathbb{Q}(v)[z_1^-, z_2^-, \dots].$$

This implies that

$$L(\Lambda) \cong \mathcal{D}(n)^- / \left(\sum_{i \in I_n} \mathcal{D}(n)^- (u_i^-)^{a_i+1} \right) \cong (\mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) / \left(\sum_{i \in I_n} \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) F_i^{a_i+1} \right)) \otimes \mathbb{Q}(v)[z_1^-, z_2^-, \dots].$$

By [34, Cor. 6.2.3], $L_0(\Lambda) \cong \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) / (\sum_{i \in I_n} \mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n) F_i^{a_i+1})$. Hence, $L_0(\Lambda)$ is the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$ -submodule of $L(\Lambda)$ generated by η_Λ and the desired decomposition is obtained.

For each family of nonnegative integers $\{m_t \mid t \geq 1\}$ satisfying all but finitely many m_t are zero, $L_0(\Lambda) \otimes \prod_{t \geq 1} (z_t^-)^{m_t}$ is a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$ -submodule of $L(\Lambda)$ since $[u_i^\pm, z_t^-] = 0$ for all $i \in I_n$ and $t \geq 1$. It is easy to see that

$$L_0(\Lambda) \otimes \prod_{t \geq 1} (z_t^-)^{m_t} \cong L_0(\Lambda - (\sum_{t \geq 1} m_t) \delta^*).$$

We conclude that

$$L(\Lambda)|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)} \cong \bigoplus_{m \geq 0} L_0(\Lambda - m \delta^*)^{\oplus p(m)}.$$

□

By [34, Th. 14.4.11], for each $\Lambda \in P^+$, the canonical basis $\{b_{\mathbf{m}}^- \mid \mathbf{m} \in \mathfrak{M}_n^{\text{ap}}\}$ of $\mathbf{U}_v^-(\widehat{\mathfrak{sl}}_n)$ gives rise to the canonical basis

$$\{b_{\mathbf{m}}^- \eta_\Lambda \neq 0 \mid \mathbf{m} \in \mathfrak{M}_n^{\text{ap}}\}$$

of $L_0(\Lambda)$. On the other hand, the crystal basis theory for the quantum generalized Kac-Moody algebra $\mathbf{U}(\widetilde{C})$ has been developed in [23]. Since all the F_i for $i \geq n$ correspond to imaginary simple roots and are central in $\widetilde{\mathbf{U}}^- = \mathcal{D}(n)^-$, applying the construction in [23, Sect. 6] shows that the set

$$\mathbf{B}' := \left\{ \left(\prod_{i \geq n} F_i^{m_i} \right) b_{\mathbf{m}}^- \mid \mathbf{m} \in \mathfrak{M}_n^{\text{ap}} \text{ and all } m_i \in \mathbb{N} \text{ but finitely many are zero} \right\}$$

forms the global crystal basis of $\widetilde{\mathbf{U}}^- = \mathcal{D}(n)^-$. We remark that \mathbf{B}' does not coincide with the canonical basis \mathbf{B}^- of $\mathcal{D}(n)^-$.

5. THE q -DEFORMED FOCK SPACE I: $\mathcal{D}(\infty)$ -MODULE

In this section we introduce the q -deformed Fock space Λ^∞ from [17] and review its module structure over $\mathcal{D}(\infty) = \mathbf{U}_v(\mathfrak{sl}_\infty)$ defined in [36, 45], as well as its $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ -module structure. We also provide a proof of [45, Prop. 5.1] by using the properties of representations of Δ_∞ . Throughout this section, we identify $\mathcal{D}(\infty)$ with $\mathbf{U}_v(\widehat{\mathfrak{sl}}_\infty)$ via taking $u_i^+ \mapsto E_i$, $u_i^- \mapsto F_i$ for all $i \in I_\infty = \mathbb{Z}$.

For each partition $\lambda \in \Pi$, let $T(\lambda)$ denote the tableau of shape λ whose box in the intersection of the i -th row and the j -th column is equipped with $j - i$ (The box is then said to be with color $j - i$). For example, if $\lambda = (4, 2, 2, 1)$, then $T(\lambda)$ has the form

-3			
-2	-1		
-1	0		
0	1	2	3

For given $i \in \mathbb{Z}$, a removable i -box of $T(\lambda)$ is by definition a box with the color i which can be removed in such a way that the new tableau has the form $T(\mu)$ for some $\mu \in \Pi$. On the contrary, an indent i -box of $T(\lambda)$ is a box with the color i which can be added to $T(\lambda)$. For $i \in \mathbb{Z}$ and $\lambda \in \Pi$, define

$$n_i(\lambda) = |\{\text{indent } i\text{-boxes of } T(\lambda)\}| - |\{\text{removable } i\text{-boxes of } T(\lambda)\}|.$$

Let \bigwedge^∞ be the $\mathbb{Q}(v)$ -vector space with basis $\{|\lambda\rangle \mid \lambda \in \Pi\}$. Following [45, 4.2], there is a left $\mathbf{U}_v(\mathfrak{sl}_\infty)$ -module structure on \bigwedge^∞ defined by

$$(5.0.2) \quad K_i \cdot |\lambda\rangle = v^{n_i(\lambda)} |\lambda\rangle, \quad E_i \cdot |\lambda\rangle = |\nu\rangle, \quad F_i \cdot |\lambda\rangle = |\mu\rangle, \quad \forall i \in \mathbb{Z}, \lambda \in \Pi,$$

where $\mu, \nu \in \Pi$ are such that $T(\mu) - T(\lambda)$ and $T(\lambda) - T(\nu)$ are a box with color i . As remarked in [36, Sect. 2], \bigwedge^∞ is isomorphic to the basic representation of $\mathbf{U}_v(\mathfrak{sl}_\infty)$ with the canonical basis $\{|\lambda\rangle \mid \lambda \in \Pi\}$.

Lemma 5.1. (1) For $i \in \mathbb{Z}$ and $\lambda, \mu \in \Pi$, if $u_i^- \cdot |\mu\rangle = |\lambda\rangle$, then there is an exact sequence

$$0 \longrightarrow S_i \longrightarrow M(\mathfrak{m}_\lambda) \longrightarrow M(\mathfrak{m}_\mu) \longrightarrow 0.$$

(2) Let $\mathfrak{m} = [i, l]$ for some $i \in \mathbb{Z}$ and $l \geq 1$. Then $\tilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle \in \mathcal{Z}|\lambda\rangle$ if $i \leq 0$ and $i + l - 1 \geq 0$ and otherwise, where $\lambda = (i + l, 1^{(-i)})$. In particular, if $i = 0$, then $\tilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle = |\lambda\rangle$.

Proof. (1) This follows directly from the definition.

(2) We proceed induction on l . The statement is trivial if $l = 1$. Suppose now $l > 1$. By the definition, $M(\mathfrak{m}) = S_i[l]$ with $\dim M(\mathfrak{m}) = \sum_{j=i}^{i+l-1} \varepsilon_j$. Then

$$u_{i+l-1}^- \cdots u_{i+1}^- u_i^- = v^{1-l} u_{\mathfrak{m}}^- + \sum_{\mathfrak{z} <_{\deg}^{\infty} \mathfrak{m}} v^{1-l} u_{\mathfrak{z}}^-.$$

For each \mathfrak{z} with $\mathfrak{z} <_{\deg}^{\infty} \mathfrak{m}$, $M(\mathfrak{z})$ is decomposable. Thus, we may write

$$M(\mathfrak{z}) = M(\mathfrak{y}) \oplus M(\mathfrak{z}_1),$$

where $\mathfrak{y} \in \mathfrak{M}_\infty$ and $\mathfrak{z}_1 = [j, i + l - j]$ for some $i < j \leq i + l - 1$. This implies that

$$u_{\mathfrak{y}}^- u_{\mathfrak{z}_1}^- = u_{\mathfrak{z}}^-.$$

By the induction hypothesis,

$$u_{\mathfrak{z}_1}^- \cdot |\emptyset\rangle \in \mathcal{Z}|\mu\rangle \text{ if } j \leq 0 \text{ and } i + l - 1 \geq 0,$$

and 0 otherwise, where $\mu = (i + l, 1^{(-j)})$. Let now $j \leq 0$ and $i + l - 1 \geq 0$ and let k_1, \dots, k_{j-i} be a permutation of $i, i + 1, \dots, j - 1$. Then

$$(u_{k_1}^- u_{k_2}^- \cdots u_{k_{j-i}}^-) \cdot |\mu\rangle = 0$$

unless $k_1 = i, k_2 = i + 1, \dots, k_{j-i} = j - 1$, and moreover

$$(u_i^- u_{i+1}^- \cdots u_{j-1}^-) \cdot |\mu\rangle = |\lambda\rangle.$$

Since $u_{\mathfrak{y}}^-$ is a \mathcal{Z} -linear combination of the monomials $u_{k_1}^- u_{k_2}^- \cdots u_{k_{j-i}}^-$, we have $\tilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle \in \mathcal{Z}|\lambda\rangle$.

Now let $i = 0$. Then $u_{\mathfrak{z}_1}^- \cdot |\emptyset\rangle = 0$ for each $\mathfrak{z}_1 = [j, i + l - j]$ with $0 < j \leq i + l - 1$. Hence,

$$\tilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle = v^{1-l} u_{\mathfrak{m}}^- \cdot |\emptyset\rangle = (u_{l-1}^- \cdots u_1^- u_0^-) \cdot |\emptyset\rangle + \sum_{\mathfrak{z} <_{\deg}^{\infty} \mathfrak{m}} u_{\mathfrak{z}}^- \cdot |\emptyset\rangle = |\lambda\rangle.$$

□

Lemma 5.2. Let $\mathfrak{m} = \sum_{l \geq 1} m_{i,l} [i, l] \in \mathfrak{M}_\infty$ and $\lambda \in \Pi$.

- (1) If there is $j \in \mathbb{Z}$ such that $\sum_{l \geq 1} m_{j,l} \geq 2$, then $\tilde{u}_{\mathfrak{m}}^- \cdot |\lambda\rangle = 0$. In particular, for each $i \in \mathbb{Z}$ and $t \geq 2$, $(u_i^-)^{(t)} \cdot |\lambda\rangle = 0$.
- (2) The element $\tilde{u}_{\mathfrak{m}}^- \cdot |\lambda\rangle$ is a \mathcal{Z} -linear combination of $|\mu\rangle$ with $\mu \in \Pi$.

Proof. (1) For each $i \in \mathbb{Z}$, we put

$$m_i = \sum_{l \geq 1} m_{i,l} \text{ and } M_i = \bigoplus_{l \geq 1} m_{i,l} S_i[l].$$

Then $M = M(\mathfrak{m}) = \bigoplus_{i \in \mathbb{Z}} M_i$, where all but finitely many M_i are zero and

$$u_{\mathfrak{m}}^- = v^{-\sum_{i > j} \langle \dim M_i, \dim M_j \rangle} (\cdots u_{[M_{-1}]}^- u_{[M_0]}^- u_{[M_1]}^- \cdots).$$

Suppose there is $j \in \mathbb{Z}$ with $m = m_j \geq 2$. Then M_j admits a decomposition

$$M_j = S_j[a_1] \oplus \cdots \oplus S_j[a_m] \text{ with } a_1 \geq \cdots \geq a_m \geq 1.$$

This implies that

$$u_{[S_j[a_m]]}^- \cdots u_{[S_j[a_1]]}^- = v^{b_j} u_{[M_j]}^-,$$

where $b_j = \sum_{1 \leq p < q \leq m} \langle \mathbf{dim} S_j[m_p], \mathbf{dim} S_j[m_q] \rangle$. Hence, it suffices to show that for each $\mu \in \Pi$,

$$u_{[M_j]}^- \cdot |\mu\rangle = v^{-b_j} (u_{[S_j[a_m]]}^- \cdots u_{[S_j[a_1]]}^-) \cdot |\mu\rangle = 0.$$

By the definition, $u_{[S_j[a_1]]}^- \cdot |\mu\rangle$ is a $\mathbb{Q}(v)$ -linear combination of ν which are obtained from μ by adding a $(j+r)$ -box for each $0 \leq r < a_1$. Thus, each such ν does not admit an indent j -box. Thus, $u_{[S_j[a_1]]}^- \cdot |\nu\rangle = 0$ and, hence, $(u_{[S_j[a_m]]}^- \cdots u_{[S_j[a_1]]}^-) \cdot |\mu\rangle = 0$. We conclude that $\tilde{u}_{\mathbf{m}}^- \cdot |\lambda\rangle = 0$.

(2) It is known that $\tilde{u}_{\mathbf{m}}^-$ is a \mathcal{Z} -linear combination of monomials of divided powers $(u_i^-)^{(t)}$ for $i \in \mathbb{Z}$ and $t \geq 1$. Since by (1), $(u_i^-)^{(t)} \cdot |\mu\rangle = 0$ for all $i \in \mathbb{Z}$, $\mu \in \Pi$ and $t \geq 2$, it follows that $\tilde{u}_{\mathbf{m}}^- \cdot |\lambda\rangle$ is then a \mathcal{Z} -linear combination of $(u_{i_1}^- \cdots u_{i_m}^-) \cdot |\lambda\rangle$, where $m = \dim M(\mathbf{m})$ and $i_1, \dots, i_m \in \mathbb{Z}$. By the definition, $(u_{i_1}^- \cdots u_{i_m}^-) \cdot |\lambda\rangle$ either is zero or lies in Π . Therefore, $\tilde{u}_{\mathbf{m}}^- \cdot |\lambda\rangle$ is a \mathcal{Z} -linear combination of $|\mu\rangle$ with $\mu \in \Pi$. \square

Proposition 5.3. (1) For each $\mathbf{m} \in \mathfrak{M}_{\infty}$,

$$\tilde{u}_{\mathbf{m}}^- \cdot |\emptyset\rangle \in \mathcal{Z}|\lambda\rangle \text{ for some } \lambda \in \Pi \text{ with } \mathbf{m}_{\lambda} \leq_{\deg}^{\infty} \mathbf{m}.$$

(2) For each $\lambda \in \Pi$,

$$\tilde{u}_{\mathbf{m}_{\lambda}}^- \cdot |\emptyset\rangle = |\lambda\rangle.$$

In particular, $b_{\mathbf{m}_{\lambda}}^- \cdot |\emptyset\rangle = |\lambda\rangle$.

Proof. (1) If $\tilde{u}_{\mathbf{m}}^- \cdot |\emptyset\rangle = 0$, there is nothing to prove. Now suppose $\tilde{u}_{\mathbf{m}}^- \cdot |\emptyset\rangle \neq 0$. By Lemma 5.2(2), we write

$$\tilde{u}_{\mathbf{m}}^- \cdot |\emptyset\rangle = \sum_{\lambda \in \Pi} f_{\lambda}(v) |\lambda\rangle,$$

where all $f_{\lambda}(v) \in \mathcal{Z}$ but finitely many are zero. If $f_{\lambda}(v) \neq 0$, then $\mathbf{dim} M(\mathbf{m}_{\lambda}) = \mathbf{dim} M(\mathbf{m})$. By Lemma 2.1(1), such a $\lambda \in \Pi$ is unique. Hence, we may suppose $\tilde{u}_{\mathbf{m}}^- \cdot |\emptyset\rangle = f(v) |\lambda\rangle$ for some $0 \neq f(v) \in \mathcal{Z}$ and $\lambda \in \Pi$. It remains to show that $\mathbf{m}_{\lambda} \leq_{\deg}^{\infty} \mathbf{m}$.

Applying Lemma 5.2(1) implies that

$$M = M(\mathbf{m}) = S_{i_1}[a_1] \oplus \cdots \oplus S_{i_t}[a_t],$$

where $i_1 < \cdots < i_t$ and $a_1, \dots, a_t \geq 1$. Then

$$u_{[S_{i_1}[a_1]]}^- \cdots u_{[S_{i_t}[a_t]]}^- = v^a u_{\mathbf{m}}^-,$$

where $a = \sum_{1 \leq p < q \leq t} \langle \mathbf{dim} S_{i_q}[a_q], \mathbf{dim} S_{i_p}[a_p] \rangle$.

We proceed induction on t to show that $M(\mathbf{m}_{\lambda}) \leq_{\deg}^{\infty} M = M(\mathbf{m})$. If $t = 1$, this follows from Lemma 5.1(2). Let now $t > 1$ and let $\mu \in \Pi$ be such that

$$(u_{[S_{i_2}[a_2]]}^- \cdots u_{[S_{i_t}[a_t]]}^-) \cdot |\emptyset\rangle = g(v) |\mu\rangle \text{ for some } 0 \neq g(v) \in \mathcal{Z}.$$

Then $u_{[S_{i_1}[a_1]]}^- \cdot |\mu\rangle = v^a f(v) g(v)^{-1} |\lambda\rangle$. By the induction hypothesis,

$$M(\mathbf{m}_{\mu}) \leq_{\deg}^{\infty} S_{i_2}[a_2] \oplus \cdots \oplus S_{i_t}[a_t].$$

By writing $u_{[S_{i_1}[a_1]]}^-$ as a \mathcal{Z} -linear combination of monomials of u_i^- 's and applying Lemma 5.1(1), there exists $X \in \text{Rep } \Delta_{\infty}$ satisfying $\mathbf{dim} X = \mathbf{dim} S_{i_1}[a_1]$ with an exact sequence

$$0 \longrightarrow X \longrightarrow M(\mathbf{m}_{\lambda}) \longrightarrow M(\mathbf{m}_{\mu}) \longrightarrow 0.$$

Since $S_{i_1}[a_1]$ is indecomposable, it follows that $X \leq_{\deg}^{\infty} S_{i_1}[a_1]$. Therefore,

$$\begin{aligned} M(\mathbf{m}_\lambda) &\leq_{\deg}^{\infty} M(\mathbf{m}_\mu) * X \leq_{\deg}^{\infty} (S_{i_2}[a_2] \oplus \cdots \oplus S_{i_t}[a_t]) * S_{i_1}[a_1] \\ &= (S_{i_2}[a_2] \oplus \cdots \oplus S_{i_t}[a_t]) \oplus S_{i_1}[a_1] = M(\mathbf{m}), \end{aligned}$$

that is, $\mathbf{m}_\lambda \leq_{\deg}^{\infty} \mathbf{m}$.

(2) Write $\lambda = (\lambda_1, \dots, \lambda_t)$ with $\lambda_1 \geq \cdots \geq \lambda_t \geq 1$. Since

$$M(\mathbf{m}_\lambda) = S_0[\lambda_1] \oplus S_{-1}[\lambda_2] \oplus \cdots \oplus S_{1-t}[\lambda_t],$$

we have that

$$u_{[S_{1-t}[\lambda_t]]}^- \cdots u_{[S_{-1}[\lambda_2]]}^- u_{[S_0[\lambda_1]]}^- = v^c u_{\mathbf{m}_\lambda}^-,$$

where

$$c = \sum_{1 \leq r < s \leq t} \langle \dim S_{1-r}[\lambda_r], \dim S_{1-s}[\lambda_s] \rangle = \sum_{1 \leq r < s \leq t} \dim \operatorname{Hom}_{\Delta_\infty}(S_{1-r}[\lambda_r], S_{1-s}[\lambda_s]).$$

By using an argument similar to that in the proof of Lemma 5.1(2), we obtain that

$$\begin{aligned} v^c u_{\mathbf{m}_\lambda}^- \cdot |\emptyset\rangle &= (u_{[S_{1-t}[\lambda_t]]}^- \cdots u_{[S_{-1}[\lambda_2]]}^- u_{[S_0[\lambda_1]]}^-) \cdot |\emptyset\rangle \\ &= v^{\lambda_1-1} (u_{[S_{1-t}[\lambda_t]]}^- \cdots u_{[S_{-1}[\lambda_2]]}^-) \cdot |(\lambda_1)\rangle \\ &= c^{\lambda_1+\lambda_2-2} (u_{[S_{1-t}[\lambda_t]]}^- \cdots u_{[S_{-2}[\lambda_3]]}^-) \cdot |(\lambda_1, \lambda_2)\rangle \\ &= v^{\lambda_1+\cdots+\lambda_t-t} |(\lambda_1, \dots, \lambda_t)\rangle = v^{\lambda_1+\cdots+\lambda_t-t} |\lambda\rangle. \end{aligned}$$

Since

$$\dim \operatorname{End}_{\Delta_\infty}(M(\mathbf{m}_\lambda)) = \sum_{1 \leq r \leq s \leq t} \dim \operatorname{Hom}_{\Delta_\infty}(S_{1-r}[\lambda_r], S_{1-s}[\lambda_s]) = c + t$$

and $\dim M(\mathbf{m}_\lambda) = \lambda_1 + \cdots + \lambda_t$, it follows that

$$\tilde{u}_{\mathbf{m}_\lambda}^- \cdot |\emptyset\rangle = v^{c+t-(\lambda_1+\cdots+\lambda_t)} u_{\mathbf{m}_\lambda}^- \cdot |\emptyset\rangle = |\lambda\rangle.$$

By (4.0.9),

$$b_{\mathbf{m}_\lambda}^- \in \tilde{u}_{\mathbf{m}_\lambda}^- + \sum_{\mathbf{p} <_{\deg} \mathbf{m}_\lambda} v^{-1} \mathbb{Z}[v^{-1}] \tilde{u}_{\mathbf{p}}^-.$$

Let $\mathbf{p} <_{\deg} \mathbf{m}_\lambda$ and suppose $\tilde{u}_{\mathbf{p}}^- \cdot |\emptyset\rangle \neq 0$. By (1), there exists $\mu \in \Pi$ with $\mathbf{m}_\mu \leq_{\deg} \mathbf{p}$ such that $\tilde{u}_{\mathbf{p}}^- \cdot |\emptyset\rangle = f(v)|\mu\rangle$ for some $f(v) \in \mathcal{Z}$. Thus, $\mathbf{m}_\mu <_{\deg} \mathbf{m}_\lambda$. By Lemma 2.1(1), $\mu = \lambda$ since $\dim M(\mathbf{m}_\mu) = \dim M(\mathbf{m}_\lambda)$. This is a contradiction. Hence, $\tilde{u}_{\mathbf{p}}^- \cdot |\emptyset\rangle = 0$. We conclude that

$$b_{\mathbf{m}_\lambda}^- \cdot |\emptyset\rangle = \tilde{u}_{\mathbf{m}_\lambda}^- \cdot |\emptyset\rangle = |\lambda\rangle.$$

□

As a consequence of the proposition above, we obtain [45, Prop. 5.1] as follows.

Corollary 5.4. *The subspace \mathcal{I} of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$ spanned by $b_{\mathbf{m}}^-$ with $\mathbf{m} \in \mathfrak{M} - \{\mathbf{m}_\lambda \mid \lambda \in \Pi\}$ is a left ideal of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$. Moreover, the map*

$$\mathbf{U}_v^-(\mathfrak{sl}_\infty)/\mathcal{I} \longrightarrow \bigwedge^\infty, \quad b_{\mathbf{m}_\lambda}^- + \mathcal{I} \longmapsto |\lambda\rangle, \quad \lambda \in \Pi$$

is an isomorphism of $\mathbf{U}_v^-(\mathfrak{sl}_\infty)$ -modules.

Proof. This follows from Proposition 5.3(2) and the fact that the set

$$\{b_{\mathbf{m}}^- \cdot |\emptyset\rangle \neq 0 \mid \mathbf{m} \in \mathfrak{M}_\infty\}$$

is a basis of \bigwedge^∞ ; see [34, Th. 14.4.11].

□

Finally, for $i \in \mathbb{Z}$ and $\lambda \in \Pi$, put

$$n_i^-(\lambda) = \sum_{j < i, j \in \bar{i}} n_j(\lambda), \quad n_i^+(\lambda) = \sum_{j > i, j \in \bar{i}} n_j(\lambda), \quad \text{and} \quad n_{\bar{i}}(\lambda) = \sum_{j \in \bar{i}} n_j(\lambda).$$

By [17, 36], there is a $\mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ -module structure on \bigwedge^∞ defined by

(5.4.1)

$$K_{\bar{i}} \cdot |\lambda\rangle = v^{n_{\bar{i}}(\lambda)} |\lambda\rangle, \quad E_{\bar{i}} \cdot |\lambda\rangle = \sum_{j \in \bar{i}} v^{n_j^-(\lambda)} E_j \cdot |\lambda\rangle, \quad F_{\bar{i}} \cdot |\lambda\rangle = \sum_{j \in \bar{i}} v^{-n_j^+(\lambda)} F_j \cdot |\lambda\rangle,$$

where $\bar{i} \in I_n = \mathbb{Z}/n\mathbb{Z}$.

6. THE q -DEFORMED FOCK SPACE II: $\mathcal{D}(n)$ -MODULE

In this section we first recall the left $\mathcal{D}(n)^{\leq 0}$ -module structure on the Fock space \bigwedge^∞ defined by Varagnolo and Vasserot in [45] and then extend their construction to obtain a $\mathcal{D}(n)$ -module structure on \bigwedge^∞ .

For each $x = \sum_{\mathbf{m}} x_{\mathbf{m}} u_{\mathbf{m}} \in \mathcal{H}(\Delta)$ with $\Delta = \Delta_n$ or Δ_∞ , we write

$$x^\pm = \sum_{\mathbf{m}} x_{\mathbf{m}} u_{\mathbf{m}}^\pm \in \mathcal{D}(\Delta)^\pm.$$

Then for each $\mathbf{d} \in \mathbb{N}I_\infty$, the map $\gamma_{\mathbf{d}} : \mathcal{H}(\Delta_n)_{\bar{\mathbf{d}}} \rightarrow \mathcal{H}(\Delta_\infty)_{\mathbf{d}}$ defined in Section 3 induces $\mathbb{Q}(v)$ -linear maps

$$\gamma_{\mathbf{d}}^\pm : \mathcal{D}(n)_{\bar{\mathbf{d}}}^\pm \longrightarrow \mathcal{D}(\infty)_{\mathbf{d}}^\pm$$

such that $\gamma_{\mathbf{d}}^\pm(x^\pm) = (\gamma_{\mathbf{d}}(x))^\pm$ for each $x \in \mathcal{H}(\Delta_\infty)$.

Following [45, 6.2], for each $\bar{i} \in I_n = \mathbb{Z}/n\mathbb{Z}$, $\lambda \in \mathfrak{M}_n$ and $x \in \mathcal{D}(n)_\alpha^-$, define

$$(6.0.2) \quad K_{\bar{i}} \cdot |\lambda\rangle = v^{n_{\bar{i}}(\lambda)} |\lambda\rangle \quad \text{and} \quad x \cdot |\lambda\rangle = \sum_{\mathbf{d}} (\gamma_{\mathbf{d}}^-(x) K_{-\mathbf{d}'}) \cdot |\lambda\rangle,$$

where the sum is taken over all $\mathbf{d} \in \mathbb{N}I_\infty$ such that $\bar{\mathbf{d}} = \alpha$ and $\mathbf{d}' = \sum_{i > j, \bar{i} = \bar{j}} d_j \varepsilon_i$. By [45, Cor. 6.2], this defines a left $\mathcal{D}(n)^{\leq 0}$ -module structure on \bigwedge^∞ which extends the Hayashi action of $\mathbf{U}_v^{\leq 0}(\widehat{\mathfrak{sl}}_n)$ on \bigwedge^∞ defined in (5.4.1).

Dually, for each $\lambda \in \Pi$ and $x \in \mathcal{D}(n)_\alpha^+$, define

$$(6.0.3) \quad x \cdot |\lambda\rangle = \sum_{\mathbf{d}} (\gamma_{\mathbf{d}}^+(x) K_{\mathbf{d}''}) \cdot |\lambda\rangle,$$

where the sum is taken over all $\mathbf{d} \in \mathbb{N}I_\infty$ such that $\bar{\mathbf{d}} = \alpha$ and $\mathbf{d}'' = \sum_{i < j, \bar{i} = \bar{j}} d_j \varepsilon_i$.

Proposition 6.1. *The formula (6.0.3) defines a left $\mathcal{D}(n)^{\geq 0}$ -module structure on \bigwedge^∞ which extends the Hayashi action of $\mathbf{U}_v^{\geq 0}(\widehat{\mathfrak{sl}}_n)$ on \bigwedge^∞ .*

Proof. Let $x \in \mathcal{D}(n)_\alpha^+$ and $y \in \mathcal{D}(n)_\beta^+$, where $\alpha, \beta \in \mathbb{N}I_n$. By the definition, we have, on the one hand, that

$$(xy) \cdot |\lambda\rangle = \sum_{\mathbf{d}} (\gamma_{\mathbf{d}}^+(xy) K_{\mathbf{d}''}) \cdot |\lambda\rangle$$

and, on the other hand, that

$$x \cdot (y \cdot |\lambda\rangle) = \sum_{\mathbf{a}, \mathbf{b}} (\gamma_{\mathbf{a}}^+(x) K_{\mathbf{a}''} \gamma_{\mathbf{b}}^+(y) K_{\mathbf{b}''}) \cdot |\lambda\rangle,$$

where the sum is taken over all $\mathbf{a}, \mathbf{b} \in \mathbb{N}I_\infty$ such that $\bar{\mathbf{a}} = \alpha$ and $\bar{\mathbf{b}} = \beta$.

Since $K_{\mathbf{a}''}\gamma_{\mathbf{b}}^+(y) = v^{(\mathbf{a}'', \mathbf{b})}\gamma_{\mathbf{b}}^+(y)K_{\mathbf{a}''}$, we obtain that

$$x \cdot (y \cdot |\lambda\rangle) = \sum_{\mathbf{d}} \sum_{\mathbf{a}+\mathbf{b}=\mathbf{d}} v^{(\mathbf{a}'', \mathbf{b})} (\gamma_{\mathbf{a}}^+(x)\gamma_{\mathbf{b}}^+(y)K_{\mathbf{d}''}) \cdot |\lambda\rangle.$$

By the definition,

$$(\mathbf{a}'', \mathbf{b}) = \left(\sum_{i < j, \bar{i}=\bar{j}} a_j \varepsilon_i, \sum_i b_i \varepsilon_i \right) = \sum_{i < j, \bar{i}=\bar{j}} b_i (2a_j - a_{j-1} - a_{j+1}) = \kappa(\mathbf{a}, \mathbf{b}).$$

Applying Lemma 3.3(2) gives that

$$(xy) \cdot |\lambda\rangle = x \cdot (y \cdot |\lambda\rangle).$$

Hence, \bigwedge^∞ becomes a left $\mathcal{D}(n)^{\geq 0}$ -module.

For each $\bar{i} \in I_n = \mathbb{Z}/n\mathbb{Z}$ and $\lambda \in \Pi$, we have

$$u_{\bar{i}}^+ \cdot |\lambda\rangle = \sum_{j \in \bar{i}} (u_j^+ K_{-\varepsilon_j''}) \cdot |\lambda\rangle.$$

Since $\varepsilon_j'' = \sum_{l < j, \bar{l}=\bar{j}} \varepsilon_l$ for each $j \in \bar{i}$, it follows that

$$K_{\varepsilon_j''} \cdot |\lambda\rangle = \prod_{l < j, \bar{l}=\bar{j}} K_{\varepsilon_l} \cdot |\lambda\rangle = v^{\sum_{l < j, \bar{l}=\bar{j}} n_l(\lambda)} |\lambda\rangle = v^{n_j^-(\lambda)} |\lambda\rangle.$$

This implies that

$$u_{\bar{i}}^+ \cdot |\lambda\rangle = \sum_{j \in \bar{i}} v^{n_j^-(\lambda)} u_j^+ \cdot |\lambda\rangle,$$

which coincides with the formula for $E_{\bar{i}} \cdot |\lambda\rangle$ in (5.4.1), as required. \square

The main purpose of this section is to prove that formulas (6.0.2) and (6.0.3) indeed define a $\mathcal{D}(n)$ -module structure on \bigwedge^∞ . The strategy is to pass to the semi-infinite v -wedge spaces defined in [27].

Let Ω denote the $\mathbb{Q}(v)$ -vector space with basis $\{\omega_i \mid i \in \mathbb{Z}\}$. By [6, Prop. 3.5], Ω admits a $\mathcal{D}(n)$ -module structure defined by

$$(6.1.1) \quad \begin{aligned} u_i^+ \cdot \omega_s &= \delta_{i+1, \bar{s}} \omega_{s-1}, \quad u_i^- \cdot \omega_s = \delta_{i, \bar{s}} \omega_{s+1} \\ K_i^{\pm 1} \cdot \omega_s &= v^{\pm \delta_{i, \bar{s}} \mp \delta_{i+1, \bar{s}}} \omega_s, \quad z_m^\pm \cdot \omega_s = \omega_{s \mp mn} \end{aligned}$$

for all $i \in I_n$ and $s, m \in \mathbb{Z}$ with $m \geq 1$. In particular, $K_\delta^{\pm 1} \cdot \omega_s = \omega_s$ for each $s \in \mathbb{Z}$. This is an extension of the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$ -action on Ω defined in [27, 1.1] as well as an extension of the $\mathcal{D}(n)^{\leq 0}$ -action on Ω defined in [45, 8.1]; see [6, 3.5].

For a fixed positive integer r , consider the r -fold tensor product $\Omega^{\otimes r}$ which has a basis

$$\{\omega_{\mathbf{i}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r} \mid \mathbf{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r\}.$$

The Hopf algebra structure of $\mathcal{D}(n)$ induces a $\mathcal{D}(n)$ -module structure on the r -fold tensor product $\Omega^{\otimes r}$. By (4.1.1), we have for each $t \geq 1$,

$$(6.1.2) \quad \begin{aligned} \Delta^{(r-1)}(z_t^+) &= \sum_{s=0}^{r-1} \underbrace{1 \otimes \cdots \otimes 1}_s \otimes z_t^+ \otimes \underbrace{K_{t\delta} \otimes \cdots \otimes K_{t\delta}}_{r-s-1} \quad \text{and} \\ \Delta^{(r-1)}(z_t^-) &= \sum_{s=0}^{r-1} \underbrace{K_{-t\delta} \otimes \cdots \otimes K_{-t\delta}}_s \otimes z_t^- \otimes \underbrace{1 \otimes \cdots \otimes 1}_{r-s-1}. \end{aligned}$$

This implies particularly that for each $t \geq 1$ and $\omega_{\mathbf{i}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r} \in \Omega^{\otimes r}$,

$$(6.1.3) \quad z_t^{\pm} \cdot \omega_{\mathbf{i}} = \sum_{s=1}^r \omega_{i_1} \otimes \cdots \otimes \omega_{i_{s-1}} \otimes \omega_{i_s \mp tn} \otimes \omega_{i_{s+1}} \otimes \cdots \otimes \omega_{i_r}.$$

By (4.0.3) and (4.0.5), for each $\alpha \in \mathbb{N}I_n$, we have

$$(6.1.4) \quad \begin{aligned} \Delta^{(r-1)}(\tilde{u}_{\alpha}^{+}) &= \sum_{\alpha=\alpha^{(1)}+\cdots+\alpha^{(r)}} v^{\sum_{s>t} \langle \alpha^{(s)}, \alpha^{(t)} \rangle} \times \\ &\quad \tilde{u}_{\alpha^{(1)}}^{+} \otimes \tilde{u}_{\alpha^{(2)}}^{+} K_{\alpha^{(1)}} \otimes \cdots \otimes \tilde{u}_{\alpha^{(r)}}^{+} K_{(\alpha^{(1)}+\alpha^{(2)}+\cdots+\alpha^{(r-1)})}, \\ \Delta^{(r-1)}(\tilde{u}_{\alpha}^{-}) &= \sum_{\alpha=\alpha^{(1)}+\cdots+\alpha^{(r)}} v^{\sum_{s>t} \langle \alpha^{(s)}, \alpha^{(t)} \rangle} \times \\ &\quad \tilde{u}_{\alpha^{(1)}}^{-} K_{-(\alpha^{(2)}+\cdots+\alpha^{(r)})} \otimes \cdots \otimes \tilde{u}_{\alpha^{(r-1)}}^{-} K_{-\alpha^{(r)}} \otimes \tilde{u}_{\alpha^{(r)}}^{-}. \end{aligned}$$

This gives the the following lemma; see [45, Lem. 8.3] and [6, Cor. 3.5.8].

Lemma 6.2. *Let $\alpha \in \mathbb{N}I_n$ and $\mathbf{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$. Then*

$$(6.2.1) \quad \tilde{u}_{\alpha}^{+} \cdot \omega_{\mathbf{i}} = \sum_{\mathbf{n}} v^{c^{+}(\mathbf{i}, \mathbf{i}-\mathbf{n})} \omega_{\mathbf{i}-\mathbf{n}},$$

where the sum is taken over the sequences $\mathbf{n} = (n_1, \dots, n_r) \in \{0, 1\}^r$ satisfying $\alpha = \sum_{s=1}^r n_s \varepsilon_{\bar{i}_s - 1}$ and

$$c^{+}(\mathbf{i}, \mathbf{i} - \mathbf{n}) = \sum_{1 \leq s < t \leq r} n_s (n_t - 1) \langle \varepsilon_{\bar{i}_t}, \varepsilon_{\bar{i}_s} \rangle;$$

$$(6.2.2) \quad \tilde{u}_{\alpha}^{-} \cdot \omega_{\mathbf{i}} = \sum_{\mathbf{n}} v^{c^{-}(\mathbf{i}, \mathbf{i}+\mathbf{n})} \omega_{\mathbf{i}+\mathbf{n}},$$

where the sum is taken over the sequences $\mathbf{n} = (n_1, \dots, n_r) \in \{0, 1\}^r$ satisfying $\alpha = \sum_{s=1}^r n_s \varepsilon_{\bar{i}_s}$ and

$$c^{-}(\mathbf{i}, \mathbf{i} + \mathbf{n}) = \sum_{1 \leq s < t \leq r} n_t (n_s - 1) \langle \varepsilon_{\bar{i}_t}, \varepsilon_{\bar{i}_s} \rangle.$$

On the other hand, let $\hat{\mathbf{H}}(r)$ be the Hecke algebra of affine symmetric group of type A which is by definition a $\mathbb{Q}(v)$ -algebra with generators T_i and X_j for $i = 1, \dots, r-1$, $j = 1, \dots, r$ and relations:

$$\begin{aligned} (T_i + 1)(T_i - v^2) &= 0, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad (|i - j| > 1), \\ X_i X_i^{-1} &= 1 = X_i^{-1} X_i, \quad X_i X_j = X_j X_i, \\ T_i X_i T_i &= v^2 X_{i+1}, \quad X_j T_i = T_i X_j \quad (j \neq i, i+1). \end{aligned}$$

This is the so-called *Bernstein presentation* of $\hat{\mathbf{H}}(r)$.

By [45, Sect. 8.2], there is a right $\hat{\mathbf{H}}(r)$ -module structure on $\Omega^{\otimes r}$ defined by

$$(6.2.3) \quad \begin{aligned} \omega_{\mathbf{i}} \cdot X_t &= \omega_{i_1} \cdots \omega_{i_{t-1}} \omega_{i_t - n} \omega_{i_{t+1}} \cdots \omega_{i_r}, \\ \omega_{\mathbf{i}} \cdot T_k &= \begin{cases} v^2 \omega_{\mathbf{i}}, & \text{if } i_k = i_{k+1}; \\ v \omega_{\mathbf{i}_{s_k}}, & \text{if } -n < i_k < i_{k+1} \leq 0; \\ v \omega_{\mathbf{i}_{s_k}} + (v^2 - 1) \omega_{\mathbf{i}}, & \text{if } -n < i_{k+1} < i_k \leq 0, \end{cases} \end{aligned}$$

where $\mathbf{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$, $\omega_{\mathbf{i}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_r}$ and

$$\omega_{\mathbf{i}_{s_k}} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_{k+1}} \otimes \omega_{i_k} \otimes \cdots \otimes \omega_{i_r}.$$

Following [45, Lem. 8.2] and [6, Prop. 3.5.5], the tensor space $\Omega^{\otimes r}$ is indeed a $\mathcal{D}(n)$ - $\widehat{\mathbf{H}}(r)$ -bimodule. Set

$$\Xi^r = \sum_{i=1}^{r-1} \text{Im}(1 + T_i) \subseteq \Omega^{\otimes r},$$

which is clearly a $\mathcal{D}(n)$ -submodule of $\Omega^{\otimes r}$. Thus, the quotient space $\Omega^{\otimes r}/\Xi^r$ becomes a $\mathcal{D}(n)$ -module. For each $\mathbf{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$, write

$$\wedge \omega_{\mathbf{i}} = \omega_{i_1} \wedge \dots \wedge \omega_{i_r} = \omega_{\mathbf{i}} + \Xi^r \in \Omega^{\otimes r}/\Xi^r.$$

By [27, Prop. 1.3], the set

$$\{\wedge \omega_{\mathbf{i}} \mid i_1 > \dots > i_r\}$$

forms a basis of $\Omega^{\otimes r}/\Xi^r$.

For each $m \in \mathbb{Z}$, let \mathcal{B}_m denote the set of sequences $\mathbf{i} = (i_1, i_2, \dots) \in \mathbb{Z}^\infty$ satisfying that $i_s = m - s + 1$ for $s \gg 0$, and set $\mathcal{B}_\infty = \cup_{m \in \mathbb{Z}} \mathcal{B}_m$. As in [45, Sect. 10.1], let Ω^∞ denote the space spanned by semi-infinite monomials

$$\omega_{\mathbf{i}} = \omega_{i_1} \otimes \omega_{i_2} \otimes \dots, \quad \text{where } \mathbf{i} = (i_1, i_2, \dots) \in \mathcal{B}_\infty.$$

Then the affine Hecke algebra $\widehat{\mathbf{H}}(\infty)$ acts on Ω^∞ via the formulas in (6.2.3). Set

$$\Xi^\infty = \sum_{i=1}^{\infty} \text{Im}(1 + T_i) \subseteq \Omega^\infty.$$

For each $\mathbf{i} = (i_1, i_2, \dots) \in \mathcal{B}_\infty$ as above, write

$$\wedge \omega_{\mathbf{i}} = \omega_{i_1} \wedge \omega_{i_2} \wedge \dots = \omega_{\mathbf{i}} + \Xi^\infty \in \Omega^\infty/\Xi^\infty.$$

By [27, Prop. 1.4], the $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$ -module structure on $\Omega^{\otimes r}/\Xi^r$ induces a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$ -module structure on Ω^∞/Ξ^∞ . Moreover, the map

$$\kappa : \bigwedge^\infty \longrightarrow \Omega^\infty/\Xi^\infty, \quad |\lambda\rangle \longmapsto \wedge \omega_{\mathbf{i}_\lambda}$$

is an injective homomorphism of $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$ -modules.

Following [27, 1.4], for each $m \in \mathbb{Z}$, write

$$|m\rangle = \omega_m \wedge \omega_{m-1} \wedge \omega_{m-2} \wedge \dots.$$

Clearly, for each $\mathbf{i} = (i_1, i_2, \dots) \in \mathcal{B}_m$, there exists a sufficiently large N such that

$$\omega_{\mathbf{i}} = (\omega_{i_1} \wedge \dots \wedge \omega_{i_N}) \wedge |m - N\rangle.$$

By [27, Lem. 2.2] and (6.1.4), for given $\alpha \in \mathbb{N}I$ and $\mathbf{i} \in \mathcal{B}_m$, there is $t \gg 0$ such that

$$u_\alpha^- \cdot (\wedge \omega_{\mathbf{i}}) = (u_\alpha^- \cdot (\omega_{i_1} \wedge \dots \wedge \omega_{i_t})) \wedge |m - t\rangle.$$

Hence, the $\mathcal{D}(n)^{\leq 0}$ -module structure on $\Omega^{\otimes r}/\Xi^r$ induces a $\mathcal{D}(n)^{\leq 0}$ -module structure on Ω^∞/Ξ^∞ ; see [45, Sect. 10.1]. Moreover, by [45, Lem. 10.1], the map $\kappa : \bigwedge^\infty \rightarrow \Omega^\infty/\Xi^\infty$ is a $\mathcal{D}(n)^{\leq 0}$ -module homomorphism.

Dually, for each given $\mathbf{i} \in \mathcal{B}_m$, there is $t \gg 0$ such that

$$u_\alpha^+ \cdot (\wedge \omega_{\mathbf{i}}) = (u_\alpha^+ \cdot (\omega_{i_1} \wedge \dots \wedge \omega_{i_t})) \wedge (K_\alpha \cdot |m - t\rangle).$$

Thus, Ω^∞/Ξ^∞ becomes a left $\mathcal{D}(n)^{\geq 0}$ -module as well. We have the following result.

Proposition 6.3. *The map κ is a $\mathcal{D}(n)^{\geq 0}$ -module homomorphism.*

Proof. We need to show that for each $\lambda \in \Pi$ and $\alpha \in \mathbb{N}I_n$,

$$\kappa(\tilde{u}_\alpha^+ \cdot |\lambda\rangle) = \tilde{u}_\alpha^+(\wedge \omega_{\mathbf{i}_\lambda}).$$

For simplicity, write $\mathbf{i} = \mathbf{i}_\lambda$. By (6.0.3),

$$\tilde{u}_\alpha^+ \cdot |\lambda\rangle = \sum_{\mathbf{d}} (\gamma_{\mathbf{d}}^+(\tilde{u}_\alpha^+) K_{\mathbf{d}''}) \cdot |\lambda\rangle = \sum_{\mathbf{d}} v^{-h(\mathbf{d})} (\tilde{u}_{\mathbf{d}}^+ K_{\mathbf{d}''}) \cdot |\lambda\rangle,$$

where the sum is taken over all $\mathbf{d} \in \mathbb{N}I_\infty$ such that $\bar{\mathbf{d}} = \alpha$ and $h(\mathbf{d}) = \sum_{i < j, \bar{i} = \bar{j}} d_i(d_{j+1} - d_j)$.

For each fixed $\mathbf{d} = (d_i) \in \mathbb{N}I_\infty$ with $\bar{\mathbf{d}} = \alpha$, we have

$$\tilde{u}_{\mathbf{d}}^+ = \cdots \tilde{u}_{d_1 \varepsilon_1}^+ \tilde{u}_{d_0 \varepsilon_0}^+ \tilde{u}_{d_{-1} \varepsilon_{-1}}^+ \cdots = \prod_{i \in \mathbb{Z}} \tilde{u}_{d_i \varepsilon_i}^+.$$

By the definition, $\tilde{u}_{\mathbf{d}}^+ \cdot |\lambda\rangle \neq 0$ implies that

$$\mathbf{d} = \sum_{s \geq 1} n_s \varepsilon_{i_s - 1},$$

where $n_s \in \{0, 1\}$ for all $s \geq 1$. Moreover, if this is the case, then

$$\tilde{u}_{\mathbf{d}}^+ \cdot |\lambda\rangle = |\mu_{\mathbf{n}}\rangle,$$

where $\mathbf{n} = (n_1, n_2, \dots)$ and $\mu_{\mathbf{n}} = \mu \in \Pi$ is determined by $\mathbf{i}_\mu = \mathbf{i} - \mathbf{n}$. Therefore, for $\mathbf{d} \in \mathbb{N}I_\infty$ with $\mathbf{d} = \sum_{s \geq 1} n_s \varepsilon_{i_s - 1}$, we have that

$$\begin{aligned} K_{\mathbf{d}''} \cdot |\lambda\rangle &= \prod_{\bar{i}_s = \bar{i}_t, i_s > i_t} K_{i_t - 1}^{n_s} \cdot |\lambda\rangle = \left(\prod_{\bar{i}_s = \bar{i}_t, i_s > i_t} v^{n_s \sum_{l \in \mathbb{Z}} (\delta_{i_t - 1, i_l} - \delta_{i_t - 1, i_l - 1})} \right) \cdot |\lambda\rangle \\ &= \prod_{i_s > i_t} v^{n_s (\delta_{i_s, \bar{i}_t + 1} - \delta_{i_s, \bar{i}_t})} |\lambda\rangle = (v^{-\sum_{i_s > i_t} n_s \langle \varepsilon_{i_t}, \varepsilon_{i_s} \rangle}) |\lambda\rangle \end{aligned}$$

and

$$h(\mathbf{d}) = \sum_{i_s > i_t} -n_s n_t (\delta_{i_s, \bar{i}_t} - \delta_{i_s, \bar{i}_t + 1}) = - \sum_{i_s > i_t} n_s n_t \langle \varepsilon_{i_t}, \varepsilon_{i_s} \rangle.$$

Since $i_s > i_t$ if and only if $s < t$, we conclude that

$$\tilde{u}_\alpha^+ \cdot |\lambda\rangle = \sum_{\mathbf{n}} v^{x(\mathbf{n})} |\mu_{\mathbf{n}}\rangle,$$

where the sum is taken over the sequences $\mathbf{n} = (n_1, n_2, \dots) \in \{0, 1\}^\infty$ satisfying $\alpha = \sum_{s=1}^r n_s \varepsilon_{i_s - 1}$ and

$$x(\mathbf{n}) = \sum_{1 \leq s < t} n_s (n_t - 1) \langle \varepsilon_{i_t}, \varepsilon_{i_s} \rangle.$$

This together with (6.2.1) implies that

$$\kappa(\tilde{u}_\alpha^+ \cdot |\lambda\rangle) = \sum_{\mathbf{n}} v^{x(\mathbf{n})} \kappa(|\mu_{\mathbf{n}}\rangle) = \sum_{\mathbf{n}} v^{x(\mathbf{n})} \wedge \omega_{\mathbf{i} - \mathbf{n}} = \tilde{u}_\alpha^+(\wedge \omega_{\mathbf{i}}) = \tilde{u}_\alpha^+(\kappa(|\lambda\rangle)).$$

This finishes the proof. \square

As a consequence of the results above, to prove that the formulas (6.0.2) and (6.0.3) define a $\mathcal{D}(n)$ -module structure on \bigwedge^∞ , it suffices to show that the $\mathcal{D}(n)^{\leq 0}$ -module and $\mathcal{D}(n)^{\geq 0}$ -module structures on $\Omega^\infty / \Xi^\infty$ define a $\mathcal{D}(n)$ -module structure. In other words, we need to show that the actions of $K_i^{\pm 1}, u_i^+, u_i^-$ ($i \in I_n$) and $\mathbf{z}_s^+, \mathbf{z}_s^-$ ($s \geq 1$) on $\Omega^\infty / \Xi^\infty$ satisfy the relations (DH1)–(DH5) in Section 4. In the following we only check the relations

$$[\mathbf{z}_t^+, \mathbf{z}_s^-] = \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2} (K_{t\delta} - K_{-t\delta}).$$

The other relations either follow from [27] or can be checked directly.

By [27, §2], for each $t \geq 1$, there are Heisenberg operators $B_t^\pm : \Omega^\infty / \Xi^\infty \rightarrow \Omega^\infty / \Xi^\infty$ taking

$$B_t(\wedge \omega_{\mathbf{i}}) \mapsto \sum_{s=1}^{\infty} \wedge \omega_{\mathbf{i} \mp t \mathbf{e}_s},$$

where $\mathbf{i} \in \mathcal{B}_\infty$ and $\mathbf{e}_s = (\delta_{i,s})_{i \geq 1} \in \mathbb{Z}^\infty$. Note that for each $\mathbf{i} \in \mathcal{B}_\infty$, $\wedge \omega_{\mathbf{i} \mp t \mathbf{e}_s} = 0$ for $s \gg 0$.

Proposition 6.4. *For each $t \geq 1$ and $\mathbf{i} \in \mathcal{B}_\infty$,*

$$B_t^+(\wedge \omega_{\mathbf{i}}) = v^t \mathbf{z}_t^+ \cdot (\wedge \omega_{\mathbf{i}}) \quad \text{and} \quad B_t^-(\wedge \omega_{\mathbf{i}}) = \mathbf{z}_t^- \cdot (\wedge \omega_{\mathbf{i}}).$$

Proof. As in [27, (49)], for each $m \in \mathbb{Z}$, write

$$|m\rangle = \omega_m \wedge \omega_{m-1} \wedge \omega_{m-2} \wedge \cdots \in \Omega^\infty / \Xi^\infty.$$

Then $\mathbf{z}_t^+ \cdot |m\rangle = 0$ and $K_\delta \cdot |m\rangle = q|m\rangle$. Write

$$\wedge \omega_{\mathbf{i}} = \omega_{i_1} \wedge \cdots \wedge \omega_{i_N} \wedge |N - m\rangle.$$

Applying (6.1.2) gives that

$$\begin{aligned} & \mathbf{z}_t^+ \cdot (\wedge \omega_{\mathbf{i}}) \\ &= \sum_{s=0}^N \underbrace{\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}}_s \wedge \mathbf{z}_t^+ \cdot \omega_{i_{s+1}} \wedge \underbrace{K_{t\delta} \cdot \omega_{i_{s+2}} \wedge \cdots \wedge K_{t\delta} \cdot \omega_{i_N}}_{N-s-1} \wedge K_{t\delta} \cdot |N - m\rangle \\ &= \sum_{s=0}^N v^t \underbrace{\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}}_s \wedge \omega_{i_{s+1} + tn} \wedge \underbrace{\omega_{i_{s+2}} \wedge \cdots \wedge \omega_{i_N}}_{N-s-1} \wedge |N - m\rangle \\ &= v^t B_t^+(\wedge \omega_{\mathbf{i}}) \quad (\text{since } B_t^+ (|N - m\rangle) = 0), \end{aligned}$$

that is, $B_t^+(\wedge \omega_{\mathbf{i}}) = v^t \mathbf{z}_t^+ \cdot (\wedge \omega_{\mathbf{i}})$. The second equality can be proved similarly. \square

Corollary 6.5. *Let $t, s \geq 1$. Then for each $\mathbf{i} \in \mathcal{B}_\infty$,*

$$[\mathbf{z}_t^+, \mathbf{z}_s^-] \cdot (\wedge \omega_{\mathbf{i}}) = \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2} (K_{t\delta} - K_{-t\delta}) \cdot (\wedge \omega_{\mathbf{i}}).$$

Proof. By [27, Prop. 2.2 & 2.6] (with $q = v$),

$$[B_t^+, B_s^-] = \delta_{t,s} \frac{t(1 - v^{2tn})}{1 - v^{2n}}.$$

This together with Proposition 6.4 implies that for each $\mathbf{i} \in \mathcal{B}_\infty$,

$$[\mathbf{z}_t^+, \mathbf{z}_s^-] \cdot (\wedge \omega_{\mathbf{i}}) = v^t [B_t^+, B_s^-] \delta_{t,s} \cdot (\wedge \omega_{\mathbf{i}}) = \delta_{t,s} \frac{tv^t(1 - v^{2tn})}{1 - v^{2n}} (\wedge \omega_{\mathbf{i}}).$$

On the other hand,

$$\begin{aligned} \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2} (K_{t\delta} - K_{-t\delta}) \cdot (\wedge \omega_{\mathbf{i}}) &= \delta_{t,s} \frac{t(v^{2tn} - 1)}{(v^t - v^{-t})^2} (v^t - v^{-t}) (\wedge \omega_{\mathbf{i}}) \\ &= \delta_{t,s} \frac{tv^t(1 - v^{2tn})}{1 - v^{2n}} (\wedge \omega_{\mathbf{i}}). \end{aligned}$$

This gives the desired equality. \square

In conclusion, \bigwedge^∞ becomes a $\mathcal{D}(n)$ -module which is obtained by the restriction of the $\mathcal{D}(n)$ -module structure on $\Omega^\infty / \Xi^\infty$ via the map κ .

7. AN ISOMORPHISM FROM $L(\Lambda_0)$ TO \bigwedge^∞

In this section we show that the Fock space \bigwedge^∞ as a $\mathcal{D}(n)$ -module is isomorphic to the basic representation $L(\Lambda_0)$ defined in Section 5. As an application, the decomposition of $L(\Lambda_0)$ in Corollary 4.5 induces the Kashiwara–Miwa–Stern decomposition of \bigwedge^∞ in [27].

Proposition 7.1. *For each $\mathfrak{m} \in \mathfrak{M}_n$, $\tilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle$ is a \mathcal{Z} -linear combination of those $|\mu\rangle$ satisfying $\mathfrak{m}_\mu \leq_{\deg} \mathfrak{m}$.*

Proof. By (6.0.2),

$$\tilde{u}_{\mathfrak{m}}^- \cdot |\emptyset\rangle = \sum_{\mathbf{d}} (\gamma_{\mathbf{d}}^-(\tilde{u}_{\mathfrak{m}}^-) K_{-\mathbf{d}'} \cdot |\emptyset\rangle), \quad \text{where } \mathbf{d}' = \sum_i \left(\sum_{j < i, \bar{j} = \bar{i}} d_j \right) \varepsilon_i.$$

Since $K_i \cdot |\emptyset\rangle = v^{\delta_{i,0}} |\emptyset\rangle$ for $i \in \mathbb{Z}$, it follows that $K_{-\mathbf{d}'} \cdot |\emptyset\rangle = v^{-\sum_{j < 0, \bar{j} = \bar{0}} d_j} |\emptyset\rangle$. By Proposition 3.4,

$$\gamma_{\mathbf{d}}^-(\tilde{u}_{\mathfrak{m}}^-) \in \sum_{\mathfrak{z}} \mathcal{Z} \tilde{u}_{\mathfrak{z}}^-,$$

where the sum is taken over $\mathfrak{z} \in \mathfrak{M}_\infty$ with $\mathcal{F}(\mathfrak{z}) \leq_{\deg}^\infty \mathfrak{m}$. Further, by Proposition 5.3(1),

$$\tilde{u}_{\mathfrak{z}}^- \cdot |\emptyset\rangle \in \mathcal{Z}|\mu\rangle$$

for some $\mu \in \Pi$ with $\mathfrak{m}_\mu^\infty \leq_{\deg}^\infty \mathfrak{z}$. This implies that

$$\mathfrak{m}_\mu = \mathcal{F}(\mathfrak{m}_\mu^\infty) \leq_{\deg} \mathcal{F}(\mathfrak{z}) \leq_{\deg} \mathfrak{m}.$$

This finishes the proof. \square

For each $\mathbf{d} = (d_i) \in \mathbb{N}I_\infty$, set

$$\sigma(\mathbf{d}) = - \sum_{i < 0, \bar{i} = \bar{0}} d_i.$$

For $\lambda \in \Pi$, we write $\sigma(\lambda) = \sigma(\dim M(\mathfrak{m}_\lambda^\infty))$. The following result was proved in [45, 9.2 & 10.1]. We provide here a direct proof for completeness.

Corollary 7.2. *For each $\lambda \in \Pi$,*

$$\tilde{u}_{\mathfrak{m}_\lambda}^- \cdot |\emptyset\rangle \in |\lambda\rangle + \sum_{\mu \triangleleft \lambda} \mathcal{Z}|\mu\rangle.$$

In particular, the $\mathcal{D}(n)$ -module \bigwedge^∞ is generated by $|\emptyset\rangle$ and the set

$$\{b_{\mathfrak{m}_\lambda}^- |\emptyset\rangle \mid \lambda \in \Pi\}$$

is a basis of \bigwedge^∞ .

Proof. Applying Corollary 3.5 gives that

$$\begin{aligned} \tilde{u}_{\mathfrak{m}_\lambda}^- \cdot |\emptyset\rangle &= \sum_{\mathbf{d}} (\gamma_{\mathbf{d}}^-(\tilde{u}_{\mathfrak{m}_\lambda}^-) K_{-\mathbf{d}'} \cdot |\emptyset\rangle) = \sum_{\mathbf{d}} v^{\sigma(\mathbf{d})} \gamma_{\mathbf{d}}^-(\tilde{u}_{\mathfrak{m}_\lambda}^-) \cdot |\emptyset\rangle \\ &= \sum_{r \in \mathbb{Z}} v^{\theta(\lambda) + \sigma(\lambda)} \tilde{u}_{\tau^{rm}(\mathfrak{m}_\lambda^\infty)}^- \cdot |\emptyset\rangle + \sum_{\mathfrak{z} \in \mathfrak{M}_\infty, \mathcal{F}(\mathfrak{z}) <_{\deg} \mathfrak{m}_\lambda} f_{\lambda, \mathfrak{z}} \tilde{u}_{\mathfrak{z}}^- \cdot |\emptyset\rangle, \end{aligned}$$

where $f_{\lambda, \mathfrak{z}} \in \mathcal{Z}$. By Proposition 5.3 and its proof,

$$\tilde{u}_{\mathfrak{m}_\lambda^\infty}^- \cdot |\emptyset\rangle = |\lambda\rangle \quad \text{and} \quad \tilde{u}_{\tau^{rm}(\mathfrak{m}_\lambda^\infty)}^- \cdot |\emptyset\rangle = 0 \quad \text{for } r > 0.$$

Furthermore, for each $r < 0$, $\tilde{u}_{\tau^{rm}(\mathfrak{m}_\lambda^\infty)}^- \cdot |\emptyset\rangle \in \mathcal{Z}|\mu\rangle$ such that $\mathfrak{m}_\mu^\infty \leq_{\deg}^\infty \tau^{rm}(\mathfrak{m}_\lambda^\infty)$. Then $\mathfrak{m}_\mu = \mathcal{F}(\mathfrak{m}_\mu^\infty) \leq_{\deg} \mathcal{F}(\tau^{rm}(\mathfrak{m}_\lambda^\infty)) = \mathfrak{m}_\lambda$, which implies that $\mu \trianglelefteq \lambda$. Since $M(\tau^{rm}(\mathfrak{m}_\lambda^\infty))$ does not have a

composition factor isomorphic to S_{λ_1-1} , μ does not contain a box with color $\lambda_1 - 1$. Thus, $\mu \neq \lambda$ and $\mu \triangleleft \lambda$.

Finally, by Proposition 7.1, for each $\mathfrak{z} \in \mathfrak{M}_\infty$ with $\mathcal{F}(\mathfrak{z}) <_{\deg} \mathfrak{m}_\lambda$, $\tilde{u}_\mathfrak{z}^- \cdot |\emptyset\rangle$ is a \mathcal{Z} -linear combination of $|\mu\rangle$ satisfying $\mathfrak{m}_\mu \leq_{\deg} \mathcal{F}(\mathfrak{z})$. Thus, $\mathfrak{m}_\mu \leq_{\deg} \mathcal{F}(\mathfrak{z}) <_{\deg} \mathfrak{m}_\lambda$, which by Lemma 2.1 implies that $\mu \triangleleft \lambda$. Hence, each $\tilde{u}_\mathfrak{z}^- \cdot |\emptyset\rangle$ is a \mathcal{Z} -linear combination of $|\mu\rangle$ with $\mu \triangleleft \lambda$. Consequently,

$$\tilde{u}_{\mathfrak{m}_\lambda}^- \cdot |\emptyset\rangle \in v^{\theta(\lambda)+\sigma(\lambda)}|\lambda\rangle + \sum_{\mu \triangleleft \lambda} \mathcal{Z}|\mu\rangle.$$

Therefore, it remains to show that

$$\theta(\lambda) + \sigma(\lambda) = 0.$$

Write $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_1 \geq \dots \geq \lambda_m \geq 1$ and set $|\lambda| = \sum_{s=1}^m \lambda_s$. We proceed induction on $|\lambda|$ to show that $\theta(\lambda) + \sigma(\lambda) = 0$. By the definition,

$$\theta(\lambda) = \sum_{s < t} \kappa(\mathbf{d}_s, \mathbf{d}_t) - \sum_{s=1}^{\ell} h(\mathbf{d}_s),$$

where $\ell = \lambda_1$ is the Loewy length of $M = M(\mathfrak{m}_\lambda^\infty)$ and $S_{\mathbf{d}_s} \cong \text{rad}^{s-1}M / \text{rad}^s M$ for $1 \leq s \leq \ell$. Let $1 \leq t \leq m$ be such that $\lambda_1 = \dots = \lambda_t > \lambda_{t+1}$ and define

$$\lambda' = (\lambda_1, \dots, \lambda_{t-1}, \lambda_t - 1, \lambda_{t+1}, \lambda_m).$$

Then $|\lambda'| = |\lambda| - 1$. By the induction hypothesis, we have $\theta(\lambda') + \sigma(\lambda') = 0$.

For each $1 \leq s \leq \ell$, let $\mathbf{d}'_s \in \mathbb{N}I_\infty$ be defined by setting $S_{\mathbf{d}'_s} \cong \text{rad}^{s-1}M' / \text{rad}^s M'$, where $M' = M(\mathfrak{m}_{\lambda'}^\infty)$. Then

$$\mathbf{d}'_\ell = \mathbf{d}_\ell - \varepsilon_{\ell-t} \quad \text{and} \quad \mathbf{d}'_s = \mathbf{d}_s \quad \text{for } 1 \leq s < \ell.$$

This implies that

$$\begin{aligned} \sum_{s=1}^{\ell} h(\mathbf{d}_s) - \sum_{s=1}^{\ell} h(\mathbf{d}'_s) &= h(\mathbf{d}_\ell) - h(\mathbf{d}'_\ell) = -\delta_{\bar{t}, \bar{1}} \quad \text{and} \\ \sum_{s < t} \kappa(\mathbf{d}_s, \mathbf{d}_t) - \sum_{s < t} \kappa(\mathbf{d}'_s, \mathbf{d}'_t) &= \sum_{1 \leq s < \ell} \kappa(\mathbf{d}_s, \varepsilon_{\ell-t}). \end{aligned}$$

Hence,

$$\theta(\lambda) - \theta(\lambda') = \sum_{1 \leq s < \ell} \kappa(\mathbf{d}_s, \varepsilon_{\ell-t}) + \delta_{\bar{t}, \bar{1}}.$$

On the other hand, $\sigma(\lambda) = \sigma(\lambda') - 1$ if $\ell - t < 0$ and $\bar{\ell} = \bar{t}$, and $\sigma(\lambda) = \sigma(\lambda')$ otherwise. A direct calculation shows that if $\ell - t \geq 0$, then

$$\sum_{1 \leq s < \ell} \kappa(\mathbf{d}_s, \varepsilon_{\ell-t}) = -\delta_{\bar{t}, \bar{1}},$$

and if $\ell - t < 0$, then

$$\sum_{1 \leq s < \ell} \kappa(\mathbf{d}_s, \varepsilon_{\ell-t}) = \begin{cases} \delta_{\bar{\ell}, \bar{t}} - 1, & \text{if } \bar{t} = \bar{1}; \\ \delta_{\bar{\ell}, \bar{t}}, & \text{if } \bar{t} \neq \bar{1}. \end{cases}$$

We conclude that in all cases,

$$\theta(\lambda) + \sigma(\lambda) = \theta(\lambda') + \sigma(\lambda') = 0.$$

□

By the definition, for each $i \in I_n = \mathbb{Z}/n\mathbb{Z}$,

$$K_i|\emptyset\rangle = v^{\delta_{i,0}}|\emptyset\rangle.$$

This together with the corollary above implies that \bigwedge^∞ is a highest weight $\mathcal{D}(n)$ -module of highest weight Λ_0 . Consequently, there is a unique surjective $\mathcal{D}(n)$ -module homomorphism

$$\varphi : \mathcal{D}(n)^- = M(\Lambda_0) \longrightarrow \bigwedge^\infty, \quad \eta_{\Lambda_0} \longmapsto |\emptyset\rangle.$$

Theorem 7.3. *The homomorphism φ induces an isomorphism of $\mathcal{D}(n)$ -modules*

$$\bar{\varphi} : L(\Lambda_0) \longrightarrow \bigwedge^\infty.$$

Proof. By definition, we have

$$F_i \cdot |\emptyset\rangle = 0 \text{ for } i \in I_n \setminus \{0\} \text{ and } F_0^2 \cdot |\emptyset\rangle = 0.$$

Therefore, φ induces a surjective homomorphism

$$\bar{\varphi} : L(\Lambda_0) = \mathcal{D}(n)^- / \left(\sum_{i \in I_n} \mathcal{D}(n)^- F_i^{\Lambda_0(h_i)+1} \right) \longrightarrow \bigwedge^\infty.$$

Since $L(\Lambda_0)$ is simple, we conclude that $\bar{\varphi}$ is an isomorphism. \square

Combining the theorem with Corollary 4.5 gives the decomposition of \bigwedge^∞ obtained by Kashiwara, Miwa and Stern in [27, Prop. 2.3].

Corollary 7.4. *As a $\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)$ -module, \bigwedge^∞ has a decomposition*

$$\bigwedge^\infty|_{\mathbf{U}'_v(\widehat{\mathfrak{sl}}_n)} \cong \bigoplus_{m \geq 0} L_0(\Lambda_0 - m\delta^*)^{\oplus p(m)}.$$

8. THE CANONICAL BASIS FOR \bigwedge^∞

In this section we show that the canonical basis of \bigwedge^∞ defined in [29] can be constructed by using the monomial basis of the Ringel–Hall algebra of Δ_n given in [8]. We also interpret the “ladder method” in [28] in terms of generic extensions defined in Section 2.

Recall that there is a bar-involution $a \mapsto \iota(a) = \bar{a}$ on $\mathcal{D}(n)^-$ which takes $\bar{v} \mapsto v^{-1}$ and fixes all \tilde{u}_α^- for $\alpha \in \mathbb{N}I_n$. Then it induces a semilinear involution on the basic representation $L(\Lambda_0)$ by setting

$$\overline{a\eta_{\Lambda_0}} = \bar{a}\eta_{\Lambda_0} \text{ for all } a \in \mathcal{D}(n)^-.$$

On the other hand, by [29], there is a semilinear involution $x \mapsto \bar{x}$ on \bigwedge^∞ which, by [45], satisfies

- (i) $\overline{|\emptyset\rangle} = |\emptyset\rangle$,
- (ii) $\overline{ax} = \bar{a}\bar{x}$ for all $a \in \mathcal{D}(n)^-$ and $x \in \bigwedge^\infty$.

Therefore, the isomorphism $L(\Lambda_0) \rightarrow \bigwedge^\infty$ given in Theorem 7.3 is compatible with the bar-involutions.

It is proved in [29, Th. 3.3] that for each $\lambda \in \Pi$,

$$(8.0.1) \quad \overline{|\lambda\rangle} = |\lambda\rangle + \sum_{\mu \triangleleft \lambda} a_{\mu,\lambda} |\mu\rangle, \text{ where } a_{\mu,\lambda} \in \mathcal{Z}.$$

Then applying the standard linear algebra method to the basis $\{|\lambda\rangle \mid \lambda \in \Pi\}$ in [31] (or see [11] for more details) gives rise to an “IC basis” $\{b_\lambda \mid \lambda \in \Pi\}$ which is characterized by

$$\overline{b_\lambda} = b_\lambda \text{ and } b_\lambda \in |\lambda\rangle + \sum_{\mu \triangleleft \lambda} v^{-1}\mathbb{Z}[v^{-1}]|\mu\rangle,$$

The basis $\{b_\lambda \mid \lambda \in \Pi\}$ is called the *canonical basis* of \bigwedge^∞ . In other words, the basis elements b_λ are uniquely determined by the polynomials $a_{\mu,\lambda}$.

Remark 8.1. Varagnolo and Vasserot [45] have conjectured that

$$b_{\mathbf{m}_\lambda}^- \cdot |\emptyset\rangle = b_\lambda \text{ for each } \lambda \in \Pi.$$

This conjecture was proved by Schiffmann [40].

In the following we provide a way to deduce (8.0.1) by using the monomial basis of the Ringel–Hall algebra of Δ_n given in [8]. As in [8, Sect. 3], set

$$I^e = I_n \cup \{\text{all sincere vectors in } \mathbb{N}I_n\}$$

and consider the set Σ of all words on the alphabet I^e . Since $\mathcal{D}(n)^-$ is isomorphic to the opposite Ringel–Hall algebra of Δ_n , we define

$$M *' N = N * M.$$

This gives the map

$$\wp^{\text{op}} : \Sigma \longrightarrow \mathfrak{M}, \quad w = \mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_t \longmapsto S_{\mathbf{a}_1} *' S_{\mathbf{a}_2} *' \cdots *' S_{\mathbf{a}_t}.$$

By [8, Sect. 9], for each $\mathbf{m} \in \mathfrak{M}$, there is a distinguished word $w_{\mathbf{m}} \in (\wp^{\text{op}})^{-1}(\mathbf{m})$ which defines a monomial $m^{(w_{\mathbf{m}})}$ on $\tilde{u}_{\mathbf{a}}^-$ with $\mathbf{a} \in \tilde{I}$ such that

$$m^{(w_{\mathbf{m}})} = \tilde{u}_{\mathbf{m}}^- + \sum_{\mathbf{p} < \deg \mathbf{m}} \theta_{\mathbf{p}, \mathbf{m}} \tilde{u}_{\mathbf{p}}^- \text{ for some } \theta_{\mathbf{p}, \mathbf{m}} \in \mathcal{Z};$$

see [8, (9.1.1)]. If $\mathbf{m} = \mathbf{m}_\lambda$ for some $\lambda \in \Pi$, we simply write $w_{\mathbf{m}_\lambda} = w_\lambda$. Thus,

$$(8.1.1) \quad m^{(w_\lambda)} = \tilde{u}_{\mathbf{m}_\lambda}^- + \sum_{\mathbf{p} < \deg \mathbf{m}_\lambda} \theta_{\mathbf{p}, \mathbf{m}_\lambda} \tilde{u}_{\mathbf{p}}^-.$$

This together with Proposition 7.1 and Corollary 7.2 implies that

$$(8.1.2) \quad m^{(w_\lambda)} |\emptyset\rangle = |\lambda\rangle + \sum_{\mu \triangleleft \lambda} \tau_{\mu, \lambda} |\mu\rangle,$$

where $\tau_{\mu, \lambda} \in \mathcal{Z}$. Since the monomials $m^{(w_\lambda)}$ are bar-invariant, we deduce that for each $\lambda \in \Pi$,

$$\overline{|\lambda\rangle} = |\lambda\rangle + \sum_{\mu \triangleleft \lambda} a'_{\mu, \lambda} |\mu\rangle \text{ for some } a'_{\mu, \lambda} \in \mathcal{Z}.$$

Comparing with (8.0.1) gives that

$$a_{\mu, \lambda} = a'_{\mu, \lambda} \text{ for all } \mu \triangleleft \lambda.$$

In case λ is n -regular, then \mathbf{m}_λ is aperiodic and the word w_λ can be chosen in Ω , the subset of all words on the alphabet $I_n = \mathbb{Z}/n\mathbb{Z}$; see [8, Sect. 4]. In other words, $m^{(w_\lambda)}$ is a monomial of the divided powers $(u_i^-)^{(t)} = F_i^{(t)}$ for $i \in I_n$ and $t \geq 1$. We now interpret the “ladder method” in [28, Sect. 6] in terms of the generic extension map. Let $\lambda = (\lambda_1, \dots, \lambda_t) \in \Pi$ be n -regular. Recall the corresponding nilpotent representation

$$M(\mathbf{m}_\lambda) = \bigoplus_{a=1}^t S_{1-a}[\lambda_a],$$

where $1 - a$ is viewed as an element in I_n . Take $1 \leq s \leq t$ with $\lambda_1 = \dots = \lambda_s > \lambda_{s+1}$ ($\lambda_{t+1} = 0$ by convention) and let $k \geq 0$ be maximal such that

$$\lambda_{s+l(n-1)+1} = \dots = \lambda_{s+(l+1)(n-1)} \text{ and } \lambda_{s+l(n-1)} = \lambda_{s+l(n-1)+1} + 1 \text{ for } 0 \leq l \leq k-1.$$

Let $i_1 \in I$ be such that $\text{soc}(S_{1-s}[\lambda_s]) = S_{i_1}$. Then for each $a = s + l(n-1)$ with $0 \leq l \leq k$,

$$\text{soc}(S_{1-a}[\lambda_a]) = S_{i_1}.$$

Define $\mu = (\mu_1, \dots, \mu_t) \in \Pi$ by setting

$$\mu_a = \begin{cases} \lambda_a - 1, & \text{if } a = s + l(n-1) \text{ for some } 0 \leq l \leq k; \\ \lambda_a, & \text{otherwise.} \end{cases}$$

It is easy to see from the construction that μ is again n -regular. Moreover, by applying an argument similar to that in the proof of [5, Prop. 3.7],

$$(k+1)S_{i_1} *' M(\mathbf{m}_\mu) = M(\mathbf{m}_\mu) * (k+1)S_{i_1} = M(\mathbf{m}_\lambda).$$

Repeating the above process, we finally obtain a sequence i_1, \dots, i_d in I_n and positive integers $k_1 = k+1, \dots, k_d$ such that

$$(k_1 S_{i_1}) *' \dots *' (k_d S_{i_d}) = M(\mathbf{m}_\lambda).$$

In other word, the word $w_\lambda := i_1^{k_1} \dots i_d^{k_d}$ lies in $(\wp^{\text{op}})^{-1}(\mathbf{m}_\lambda)$. It can be also checked that the word w_λ is distinguished. Thus, the corresponding monomial

$$m^{(w_\lambda)} = (u_{i_1}^-)^{(k_1)} \dots (u_{i_d}^-)^{(k_d)} = F_{i_1}^{(k_1)} \dots F_{i_d}^{(k_d)}$$

gives rise to the equality (8.1.2) for the element $m^{(w_\lambda)}|\emptyset\rangle$. We remark that $m^{(w_\lambda)}|\emptyset\rangle$ coincides with the element $A(\lambda)$ constructed in [28, (8)] by using the “ladder method” of James and Kerber [22].

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